Stochastic Local Volatility models

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The aim of the project is to give the rigorous framework of stochastic local volatility models and to use it in order to expose the calibration issues. All the simulations were performed using Python.

1 Introduction

We fix the theoretical probability context for the whole report unless otherwise precised.

- Let $(\Omega,\mathscr{F},\mathbb{P})$ a complete probability space with a probability measure $\mathbb{P}$ assumed to be the asset historical measure.
- We suppose that a $\mathbb{P}$- classical brownian motion denoted $(B_t)_{t\geq 0}$ is well defined within that space.
- $\mathscr{F}$ is a complete filtration right continuous.
- We denote $\mathbb{Q}$ the risk neutral probability measure.

Unless otherwise stated, T is a fixed maturity.

2 Volatility modelling

Let's consider a one dimensional underlying asset with the following risk neutral dynamic:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma_t dW_t^{\mathbb{Q}} \\ S_0^{s_{ini}} = s \end{cases} \tag{1}$$

We can consider different scenarios :

- **Both r and σ are constant** : This is actually the toy model of Black and Scholes where european call and put options are priced by a formulas, But the implied volatility surface is flat which is not realistic compared to the market features.
- **r constant and σ deterministic time dependent** : Black and Scholes Closed formulas for european call and put options are also possible with the constant Black and Scholes volatility:

$$\sigma_{BS}^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt \tag{2}$$

- **r constant and σ stochastic** : For instance the Stein-Stein (gaussian) or the Heston (mean reverting CIR) models.
  - For affine models, we have a semi-closed european call and put pricing formulas with the inverse Fourier transformation.
  - The heavy tails distributions of assets are captured by the Heston model (the volatility square root effet) and can be refined by some specific modifications of the Heston model (Heston $\frac{3}{2}$, Heston ++,...).
  - The smile is driven by the correlation structure between the underlying and the stochastic volatility, rotation around the money depending of the correlation sign and value.
- **Both r and σ are stochastic** : For instance an hybrid Heston model with Hull and White spot interest rate. Pricing derivatives are perfomed via an Hybrid Monte carlo with approximations on the covariance spot-rate matrix.
  - Based mainly on the computation of $\mathbb{E}^{\mathbb{Q}}\left(\sqrt{v_t}\right)$ by series expansion where $(v_t)_t$ is a CIR process

Actually, there is another issue which is to mix between the local volatility models and stochastic volatility models and to build stochastic local volatility models, which are formulated as follow:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma_t \tilde{\sigma}(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ S_0^{s_{ini}} = s_{ini} \end{cases} \tag{3}$$

Where $\tilde{\sigma}$ is a time and space dependent determinitic function.

Thus, the SLV models are meant to encompass both local and stochastic volatility models.

3 The SLV model as a formulation of a McKean-Vlasov SDE

3.1 Preliminary of McKean-Vlasov stochastic differential equations

Brownian driven **McKean-Vlasov** processes or McKean-Vlasov diffusions are stochastic process which can be described by SDEs of the form:

$$\begin{cases} dX_t^x = b\left(t, X_t^x, \mu_t\right) dt + \sigma\left(t, X_t^x, \mu_t\right) dW_t^{\mathbb{P}} \\ \mu_t = \mathscr{L}\left(X_t^x\right) \end{cases} \tag{4}$$

Simply speaking, it's a process whose dynamic depend on its law.
We will focus on the unidimensionnal case.

3.2 Existence of a strong solution to the McKean-Vlasov stochastic differential equations

We first introduce these notations:

- $\mathscr{P}^2(\mathbb{R})$ the set of all probability measures ν with a finite second moment, ie : $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$

- The Wasserstein distance:

$$W_p(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu), law(X,Y)=\pi} \left(\mathbb{E}^{\mathbb{P}}\left[|X-Y|\right]^p\right)^{\frac{1}{p}} \tag{5}$$

Where $\Pi(\mu,\nu)$ is the set of probability measures whose first and second marginals are respectively μ and ν.

We have the following weak existence of $(4)^{th}$ solution.

Theorem 1 *We consider the following assumptions:*

- ***H1***:

$$\begin{cases} b : [0,T]\times\mathbb{R}\times\mathscr{P}^2(\mathbb{R}) \longrightarrow \mathbb{R} \\ \sigma : [0,T]\times\mathbb{R}\times\mathscr{P}^2(\mathbb{R}) \longrightarrow \mathbb{R} \end{cases}$$

are measurable ones and there exists a constant C such that :

$$\forall(t,x,\mu)\in[0,T]\times\mathbb{R}\times\mathscr{P}^2(\mathbb{R}), |b(t,x,\mu)|+|\sigma(t,x,\mu)| \leq C\left(1+|x|\right) \tag{6}$$

- ***H2***: *There exist an L such that for any $(t,t^{')\in[0,T]^2}$ and any $(x,\mu),(x^{'},\mu^{'})\in(\mathbb{R}\times\mathscr{P}^2(\mathbb{R}))^2$:*

$$\begin{cases} |b(t,x,\mu)-b(t,x^{'},\mu^{'})| \leq L\left(1+|x-x^{'}|-W_2(\mu,\mu^{'}),\right) \\ |\sigma(t,x,\mu)-\sigma(t,x^{'},\mu^{'})| \leq L\left(1+|x-x^{'}|-W_2(\mu,\mu^{'}),\right) \end{cases} \tag{7}$$

Under these assumptions , (4) admits a unique solution bounded in p

The following theorem determines the conditions of a strong solution of a specified formulation of (4).

Theorem 2 *We consider the following Mckean-Vlasov SDE:*

$$\begin{cases} dX_t^x = b\left(t, X_t^x, \mathbb{E}^{\mathbb{P}}\left(\phi(X_t)\right)\right)dt + \sigma\left(t, X_t^x, \mathbb{E}^{\mathbb{P}}\left(\psi(X_t)\right)\right)dW_t^{\mathbb{P}} \\ X_0^x = x \end{cases} \tag{8}$$

*Furthermore, we assume that all the function $b(t,.,.)$,$\sigma(t,.,.)$,ϕ and ψ are measurable and lipschitz continuous. under the assumption **H1**, the SDE (8) admits a unique strong solution bounded in $\mathscr{L}^p$ dor p>0.*

4 SLV models: Particular cases

As said above, the stochastic local volatiltiy models were introduceed to encompass the advantages both local and stochastic volatility models in terms of asset dynamic and smile calibration. But, as presented in formula (3) there is the local volatility component in the slv models that appears. Let's tackle this first.

4.1 Pure local volatility models

4.1.1 Pure local volatility models in non stochastic interest rates context

The idea behind local volatility models is to come up with a single factor volatility model capable of calibrating the european call and put instruments observed in the market.

The well known Dupire formula was developped :

Theorem 3 *The local volatility surface that best fits the call and put cotations is:*

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + Kr\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)} \tag{9}$$

Thus, by the identification of the call surface observed in the market we obtain a local volatility surface coherent with market data.

Proof 1 *We apply the Itô-Tanaka to the discounted call payoff:*

$e^{-rT}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-rt}1_{S_t^{s_{ini}}>K}dS_t^{s_{ini}} - \int_0^T re^{-rt}\left(S_t^{s_{ini}} - K\right)_+ dt + \frac{1}{2}\int_0^T e^{-rt}dL_t^K$
Where $(L_t^K)_{t\geq 0}$ *is the local time of the equity.*

Then:

$e^{-rT}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-rt}1_{S_t^{s_{ini}}>K}dS_t^{s_{ini}} - \int_0^T re^{-rt}\left(S_t^{s_{ini}} - K\right)_+ dt + \frac{1}{2}K^2\int_0^T e^{-rt}\sigma^2(t,K)\delta_K(S_t)dt$

We assume that a the stochastic integral in the asset's dynamic vanishes under expectation, then by Fubini theorem we obtain:

$C(T,K) = C(0,K) + \int_0^T Kr\mathbb{E}^{\mathbb{Q}}\left(e^{-rt}1_{S_t^{s_{ini}}>K}\right)dt + \frac{1}{2}K^2\int_0^T \mathbb{E}^{\mathbb{Q}}\left(e^{-rt}\sigma^2(t,K)\delta_K(S_t^{s_{ini}})\right)dt$

By differentiation with respect to maturity, we get:

$\partial_T C(T,K) = Kr\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}1_{S_T^{s_{ini}}>K}\right) + \frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}\delta_K(S_T^{s_{ini}})\right)\sigma^2(T,K)$

We conclude then that the local volatility can be expressed as follow:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + Kr\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)}$$

P.S: The proof above is unchanged if the rates are deterministic ones as they will be out of the risk neutral expectation and the little change int he local volatility surface will be:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + Kr_T\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)} \tag{10}$$

4.1.2 Pure local volatility models with stochastic interest rates

Now, we consider an hybrid local volatility equity-rate model :

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}})dW_t^{\mathbb{Q}} \\ dr_t = b(t,r_t)dt + \sigma(t,r_t)dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho_t dt \end{cases} \tag{11}$$

Ad we want to determine the local volatility that best fits the market data.
Here, there will be a significant change in the Dupire's local volatility.

Theorem 4 *The hybrid local volatility surface that best fits the call and put cotations is:*

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) - K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{12}$$

Thus, by the identification of the call surface observed in the market we obtain a local volatility surface coherent with market data.

Proof 2 *By Itô-tanaka formula:*

$$e^{-\int_0^T r_s ds}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K} dS_t^{s_{ini}} - \int_0^T r_t e^{-\int_0^t r_s ds}\left(S_t^{s_{ini}} - K\right)_+ dt + \frac{1}{2}\int_0^T e^{-\int_0^t r_s ds} dL_t^K$$

Where $(L_t^K)_{t\geq 0}$ is the local time of the equity.

Then:

$$e^{-\int_0^T r_s ds}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K} dS_t^{s_{ini}} - \int_0^T r_t e^{-\int_0^t r_s ds}\left(S_t^{s_{ini}} - K\right)_+ dt +$$

$$\frac{1}{2}K^2\int_0^T e^{-\int_0^t r_s ds}\sigma^2(t,K)\delta_K(S_t)dt$$

We assume that a the stochastic integral in the asset's dynamic vanishes under expectation, then by Fubini theorem we obtain:

$$C(T,K) = C(0,K) + \int_0^T K\mathbb{E}^{\mathbb{Q}}\left(r_t e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K}\right)dt + \frac{1}{2}K^2\int_0^T \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^t r_s ds}\sigma^2(t,K)\delta_K(S_t^{s_{ini}})\right)dt$$

By differentiation with respect to maturity, we get:

$$\partial_T C(T,K) = K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) + \frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)\sigma^2(T,K)$$

We conclude then the expression of the hybrid local volatility surface.

- The term $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)$ is nothing else but $\frac{\partial^2 C(T,K)}{\partial K^2}$
- All the terms are perfectly calibrable
- The only term that distrubs is $\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)$ which needs a special treatment.
- In the case of deterministic interest rates, the current hybrid local volatility surface turns into the local volatility surface of the formula (10).

4.1.3 Calibration of a Pure local volatility surface in practice

As market data are not sufficiently continuous in strike and maturity in order to have a good estimators of the partial derivatives $\partial_T C(T,K)$, $\partial_K C(T,K)$ and $\partial_{KK} C(T,K)$ by finite differences :

$$\partial_T C(T,K) \approx \frac{C_{BS}(T+\delta T, K, \sigma_{impl}(T+\delta T, K)) - C_{BS}(T-\delta T, K, \sigma_{impl}(T-\delta T, K))}{2\delta T} \tag{13}$$

$$\partial_K C(T,K) \approx \frac{C_{BS}(T, K+\delta K, \sigma_{impl}(T, K+\delta K)) - C_{BS}(T, K-\delta K, \sigma_{impl}(T, K-\delta K))}{2\delta K} \tag{14}$$

$$\partial_{KK}C(T,K) \approx \frac{C_{BS}(T,K+\delta K,\sigma_{impl}(T,K+\delta K)) + C_{BS}(T,K-\delta K,\sigma_{impl}(T,K-\delta K)) - 2C_{BS}(T,K,\sigma_{impl}(T,K}{\delta K^2} \tag{15}$$

Thus, an arbitrage free interpolation scheme is crucial. Its can be applied directly in the call surface or applied in the equity smile.

We denote $\mathbb{T}$ and $\mathbb{K}$ the set of all maturities and strikes available in the equity smile. We present a number of interpolation schemes.

- Linear interpolation

The linear interpolation was performed for each strike to interpolate in maturity. That is to say :

$$\forall T \in \mathbb{T}, \hat{\sigma}(T,.) = \sum_{K \in \mathbb{K}} L(T,.)\sigma(T,.) \tag{16}$$

Where L in the Lagrange interpolation polynomial function.
The ideas that were tested were to interpolate per strike and maturity the total variance $T \longrightarrow T\sigma^2(T,.)$ and $K \longrightarrow K\sigma^2(.,K)$ instead of the implied volatility as we expect that the variance is smooth enough.

- L^2 interpolation

From a smoothness point of view, we experienced Hermite polynomial interpolation rather than just a simple linear interpolation as the functional interpolated won't be C^2 in the points of the interpolation. We recall the definition of Hermite polynomial functions:

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{\frac{-x^2}{2}}}{dx^n} \tag{17}$$

Thus, the interpolated volatility function is:

$$\forall T \in \mathbb{T}, \hat{\sigma}(T,.) = \sum_{n=1}^{N} \alpha_n H_n(.) \tag{18}$$

Where $(\alpha_n)_{n \in \{1,\dots N\}}$ are determined by least square minimisation.

- Stineman interpolation

It's a monotone convex interpolation method as a functional interpolation basis a set of rational fractions.

Now that we have all the ingredients, here is calibration algorithm:

Input: Model parameters, Equity smile
Output: Local volatility surface

1. Perform interpolation of the Smile by strike and maturity
2. Estimate all the partial derivatives by finite differences
3. Agregate all the terms
4. Perform grid interpolation
5. Return the hybrid local volatility surface

The only difference to be careful with in the case of stochastic interest rates is to evaluate the hybrid term, one way is to perform a Monte Carlo method:

$$\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) \approx \frac{1}{M}\sum_{i=1}^{M}\left(r_T^{(i)} e^{-\delta \sum_{k=1}^{N} r_{t_k}^{(i)}} 1_{S_T^{(i)}>K}\right) \tag{19}$$

The calibration algorithm of the hybrid local volatility surface will be:

Input: Model parameters, Equity smile, Monte Carlo parameters
Output: Hybrid local volatility surface

1. Perform interpolation of the Smile by strike and maturity
2. Estimate all the partial derivatives by finite differences
3. Perform the Monte Carlo computation of the hybrid term
4. Agregate all the terms
5. Perform grid interpolation
6. Return the hybrid local volatility surface

4.1.4 Numerical calibration examples of the pure local volatility surface

The following results were obtained using a linear interpolation. This is the hybrid local volatility surface constructed with the following conditions:

- An Hull & White interest rate model
- Fixed rate parameters and a volatility rate of 30%

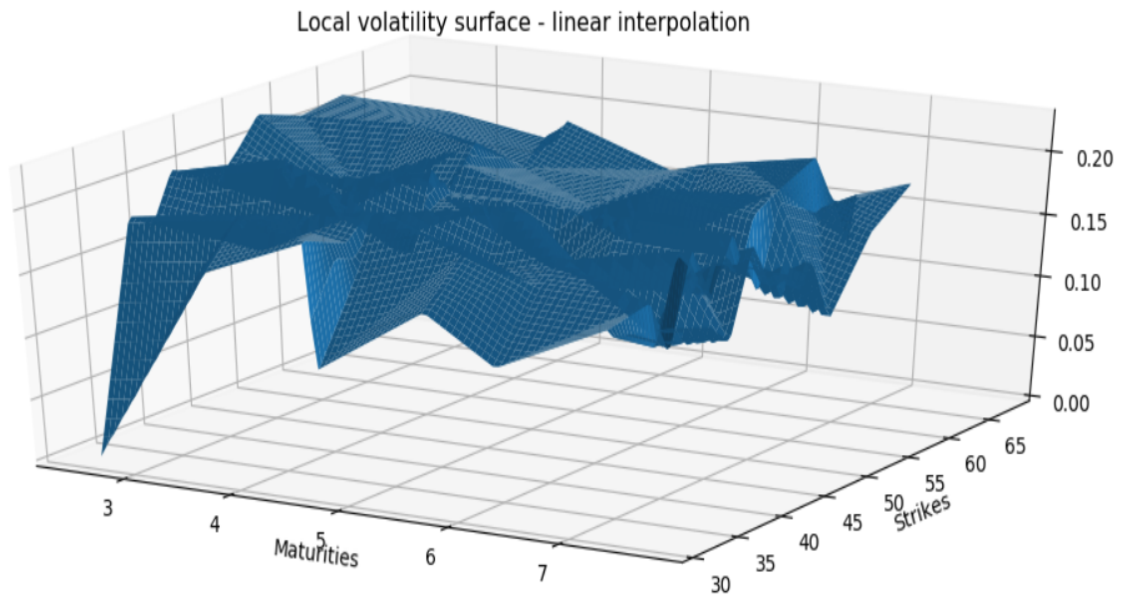

Figure 1: Hybrid local volatility surface with non flat equity smile

In order to be sure of the Hybrid local volatility calibration engine, we must test it in a specific context where the equity smile is flat equal to 30% and interest rates are constants.
This is the result :

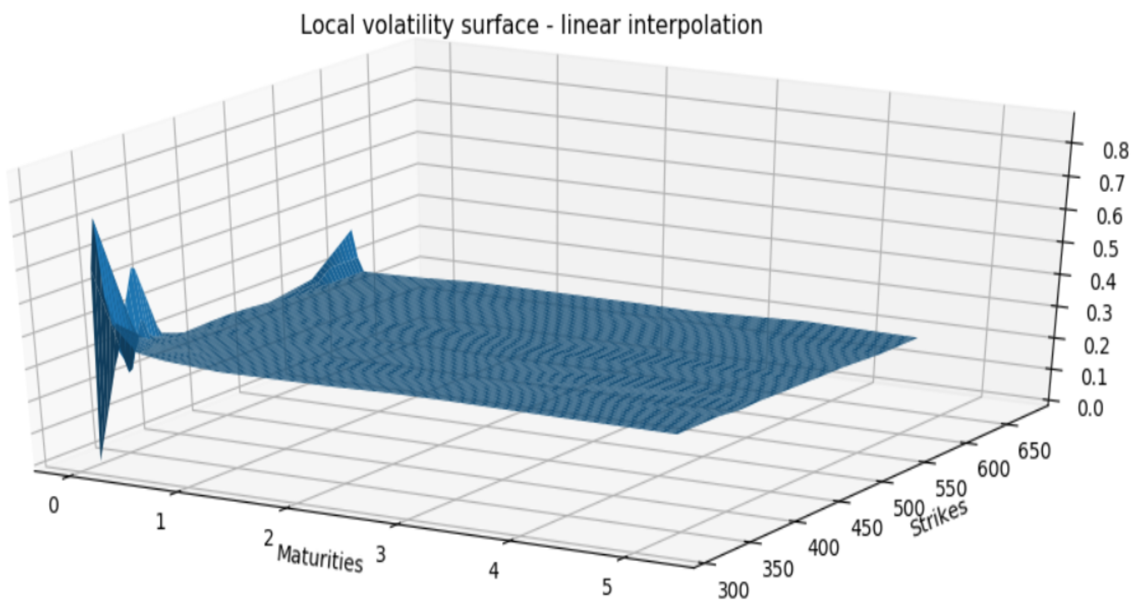

Figure 2: Hybrid local volatility surface with flat equity smile and low rate vol

And so for the Dupire's local volatility calibration engine :

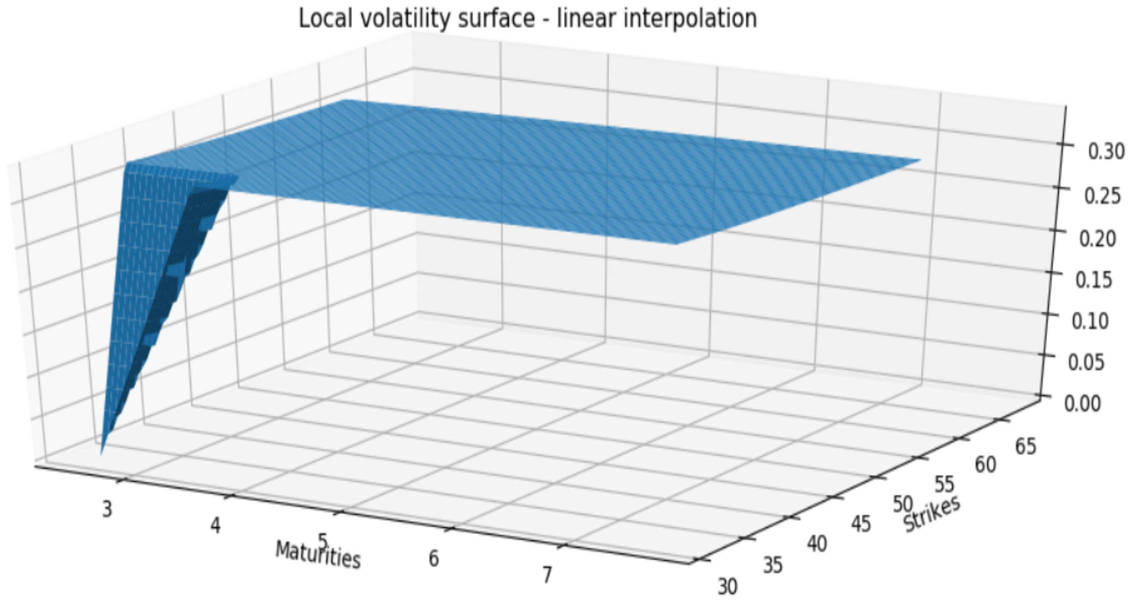

Figure 3: Dupire's local volatility surface with flat equity smile

Which is coherent as the local volatility is equal to the implied volatility in the case of a flat equity smile.(See Appendix)

5 SLV models: Full framework

5.1 The formulation

The SLV models are a mixed of local and stochastic volatility models. We recall :

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma_t \tilde{\sigma}(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ S_0^{s_{ini}} = s_{ini} \end{cases} \tag{20}$$

We first consider constant interest rates.

It's wanted from the SLV models to calibrate marlet prices as well as a local volatility model does. This induces to use the Markovian projection property.

Theorem 5 *(Gyongy) We consider a stochastic process:*

$$dX_t = b_t dt + \alpha_t dB_t^{\mathbb{Q}} \tag{21}$$

The following stochastic process :

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t^{\mathbb{Q}} \tag{22}$$

has the same marginal distribution as X with:

$$\begin{cases} b(t, x) = \mathbb{E}^{\mathbb{Q}}\left(b_t | X_t = x\right) \\ \sigma(t, x) = \sqrt{\mathbb{E}^{\mathbb{Q}}\left(\alpha_t^2 | X_t = x\right)} \end{cases} \tag{23}$$

As a result, there is a specific choice of the local volatility $\tilde{\sigma}$ that is calibrated to the market data, which is:

$$\tilde{\sigma}(t,K) = \frac{\sigma_{Dup}(t,K)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left(\sigma_t^2 | S_t^{s_{ini}} = K\right)}} \tag{24}$$

5.2 A representation of the conditional expectation

Let's consider the general case of a computation of a conditional expectation $\mathbb{E}^{\mathbb{Q}}\left(Y|X=x\right)$.

Using the Bayes formula, we have:

$$\mathbb{E}^{\mathbb{Q}}(Y/X=x) = \frac{\mathbb{E}^{\mathbb{Q}}(Y\delta_X(x))}{\mathbb{E}^{\mathbb{Q}}(\delta_X(x))} \tag{25}$$

As the dirac is not smooth enough, one way to compute (25) is by the way of kernel approximation :

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}(\hat{\delta}_X(x)) = \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) \\ \mathbb{E}^{\mathbb{Q}}(Y\hat{\delta}_X(x)) = \sum_{i=1}^{n} Y_i K\left(\frac{X_i - x}{h}\right) \\ \mathbb{E}^{\mathbb{Q}}(Y/X=x) \approx \sum_{i=1}^{n} \frac{K\left(\frac{X_i-x}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j-x}{h}\right)} Y_i \end{cases} \tag{26}$$

Where h is the bandwidth of the kernel estimation fixed to $n^{-\frac{1}{5}}$ for the numerical simulations coming afterwards.

We give here some examples of kernel functions.

- The gaussian kernel :

$$K(t) = \frac{1}{\sqrt{2\pi}} exp(-\frac{t^2}{2}) \tag{27}$$

- The Epanechnikov kernel :

$$K(t) = (1 - \frac{t^2}{h^2}) 1_{\{\frac{t}{h}<1\}} \tag{28}$$

- The triweight kernel :

$$K(t) = \frac{105}{48}\left((1-t^2)^3\right)_+ 1_{\{t<1\}} \tag{29}$$

- The normalised triweight kernel :

$$K(t) = \frac{105}{48}\left(\left(1 - \left(\frac{t}{h}\right)^2\right)^3\right)_+ 1_{\{\frac{t}{h}<1\}} \tag{30}$$

Thus, the computation of the conditional expectation of (24) will be computed using the approximation (26). But there is still the problem of the generation of (X_i) and (Y_i) which will be tackled in the particle method.

6 The particle method

6.1 Theoretical basis

The aim of the particle method is to be able to simulate a McKean-Vlasov process. The main element that is non classic comparing to a simple SDE is the law dependence. As a result, the aim is to approximate that law μ_t with the following random measure :

$$\mu_t(x) \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}(x) \tag{31}$$

Where $(X_t^i)_{i=1,\dots,N}$ can be seen as N stochastic processes that represent a system of N interacting particles. N is assumed to be high as the approximation (31) is a convergence in law formulation.

As a result, the 'effective local volatility' can be expressed in such a computing formula:

$$\tilde{\sigma}(t,K) = \sigma_{Dup}(t,K)\sqrt{\frac{\sum_{i=1}^{n}\delta_{t,N}\left(S_t^i - K\right)}{\sum_{i=1}^{n}(\sigma_t^i)^2\delta_{t,N}\left(S_t^i - K\right)}} \tag{32}$$

In order to do that we must have a strong convergence theorem that justifies the approximation (31). Here is **the propagation of chaos** convergence result.

Theorem 6 *We consider the McKean-Vlasov SDE (4).*
*On consider the hypothesis **H1** and **H2** of **theorem 1**.*
In addition, we suppose that :

- *b and σ have a polynomial growth and Lipshitz continous*
- *b and σ are $\frac{1}{2}$-Holder in time*

We have :

$$\lim_{N\longrightarrow\infty}\sup_{0<i<N}\mathbb{E}^{\mathbb{P}}\left[\sup_{0<t<T}|X_t^i - X_t^{i,N}|^2\right] = 0 \tag{33}$$

Furthermore, we can control the rate of convergence of that limit :

$$\sup_{0<i<N}\mathbb{E}^{\mathbb{P}}\left[\sup_{0<t<T}|X_t^i - X_t^{i,N}|^2\right] = O\left(\frac{1}{\sqrt{N}}\right) \tag{34}$$

As a result, this rate of convergence is similar to the one of the explicit Euler scheme.

Proof 3 *The proof use classical arguments similar to the rate of convergence of the Euler scheme :*

- *Hypothesis **H1** and **H2***
- *BDG inequality for the local martingale*
- *Gronwall inequality*

Remark 1 *As a result, we can now write the following claim for N high enough for any measurable function f with respect to the measure $d\mathbb{P}_X(dx) = p(x,t)\lambda(dx)$ in one dimension:*

$$\frac{1}{N}\sum_{i=1}^{N} f\left(X_t^i\right) \approx \int_{\mathbb{R}} f(x)p(x,t)\lambda(dx) \tag{35}$$

The previous approximation is in L^1.

6.2 The particle method algorithm applied to SLV calibration

So, the particle method can be summarised is the following algorithm:

Input: SLV Model, $(t_k)_k$ a time discretisation of [0,T], a threashold η, a calibrated Dupire's local volatility surface
Output: Hybrid local volatility surface

1. **Initialisation** k=1 :
   - Set
   $$\forall t \in [0, t_1], \tilde{\sigma}(t, K) = \frac{\sigma_{Dup}(0, K)}{\sigma_0} \tag{36}$$
2. **Iteration k**
   - Simulate an N-sample of $(S^i_{t_k}, \sigma^i_{t_k})_{i=1,...,N}$ from t_{k-1} to t_k using the SLV model
   - Find both :
   $$\begin{cases} i^k_{min} = \inf\{i \in \{1, ..., N\}, st : \delta_{t_k,N}\left(S^i_{t_k} - K\right) > \eta\} \\ i^k_{max} = \sup\{i \in \{1, ..., N\}, st : \delta_{t_k,N}\left(S^i_{t_k} - K\right) > \eta\} \end{cases} \tag{37}$$
   - Set :
   $$\tilde{\sigma}(t_k, K) = \sigma_{Dup}(t_k, K)\sqrt{\frac{\sum_{i=i^k_{min}}^{i^k_{max}} \delta_{t_k,N}\left(S^i_{t_k} - K\right)}{\sum_{i=i^k_{min}}^{i^k_{max}} (\sigma^i_{t_k})^2 \delta_{t_k,N}\left(S^i_{t_k} - K\right)}} \tag{38}$$
3. **Refresh** k=k+1

Thus, the Dirac function can be approximated by one of the regularising kernels above.

Based on (35), we can conclude some interesting implementation considerations.

Remark 2 *We notice that :*

- *The computation of the local volatility part of SLV models depend on a Dupire's local volatility surface and the whole SLV model in the previous simulation time.*
- *The computations can be performed in parallele : on the one hand the Dupire's local volatility and on the other hand the term* $\sqrt{\frac{\sum_{i=i^k_{min}}^{i^k_{max}} \delta_{t,N}\left(S^i_{t_k} - K\right)}{\sum_{i=i^k_{min}}^{i^k_{max}} (\sigma^i_{t_k})^2 \delta_{t,N}\left(S^i_{t_k} - K\right)}}$.
- *The interest rates characteristics are encompassed in* $\sigma_{Dup}(t_k, K)$.
- *The step of the formula (37) of the particle method algorithm is done in order to speed up each iteration of the algorithm. In practice, if the number of particles is not that high, the computational time of the calibration is bearable without checking the weight of the Dirac mass* $\delta_{t_k,N}\left(S^i_{t_k} - K\right)$ *with respect of the threshold* η.

6.3 Quid on the hybrid stochastic local volatility

In the case of an equity-rate hybrid SLV model, the main difference is on the local volatility considered in that case.

By definition, we consider an hybrid SLV model fomulated as such:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}})\sigma_t dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \tilde{\sigma}(t, r_t)dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho_t dt \end{cases} \tag{39}$$

We want to choose the stochastic part of the SLV model so that its markovian projection is the following model:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma_{Hyb}(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \tilde{\sigma}(t, r_t)dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho_t dt \end{cases} \tag{40}$$

Where σ_{Hyb} is the hybrid local volatility that is in the best inline with Market data, which is exactly (12).

As a consequence the chosen local volatility is defined by :

$$\tilde{\sigma}(t, K) = \frac{\sigma_{Hyb}(t, K)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left(\sigma_t^2 | S_t^{s_{ini}} = K\right)}} \tag{41}$$

With :

$$\sigma_{Hyb}^2(T, K) = \frac{\partial_T C(T, K) - K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \partial_{KK} C(T, K)} \tag{42}$$

As a result, the particle method does not change except that in the place of σ_{Dup} we have σ_{Hyb}.

7 Example of SLV models

This section is dedicated to expose some particular examples of SLV models used in practice.

7.1 Single SLV models (non hybrid)

7.1.1 Bergomi's local stochastic volatility model

Bergomi's model is based on a forward variance model as a stochastic volatility model:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = \sigma(t, S_t^{s_{ini}})\sqrt{\xi_t^T} dW_t^{\mathbb{Q}} \\ \xi_t^T = \xi_0^T f^T(t, x_t^T) \\ f^T(t, x) = exp\left(2\sigma x - 2\sigma^2 h(t, T)\right) \\ x_t^T = \alpha_\theta\left((1-\theta)\, e^{-\kappa_X (T-t)} X_t + \theta e^{-\kappa_Y (T-t)} Y_t\right) \\ \alpha_\theta = \left((1-\theta)^2 + \theta^2 + 2\rho_{X,Y}\theta\,(1-\theta)\right)^{-\frac{1}{2}} \\ X_t = \int_0^t e^{-\kappa_X (t-s)} dW_s^X \\ Y_t = \int_0^t e^{-\kappa_Y (t-s)} dW_s^Y \end{cases} \tag{43}$$

We see that X and Y are Orenstein Ulenbeck processes with mean level 0 and volatility equal to 1. Furthermore, we have the following notations:

- $h(t,T) = (1-\theta)^2 e^{-2\kappa_X(T-t)}\mathbb{E}\left(X_t^2\right) + \theta^2 e^{-2\kappa_Y(T-t)}\mathbb{E}\left(Y_t^2\right) + 2\theta\left(1-\theta\right)e^{-(\kappa_X+\kappa_Y)(T-t)}\mathbb{E}\left(X_tY_t\right)$

The computation of $\mathbb{E}\left(X_t^2\right)$, $\mathbb{E}\left(Y_t^2\right)$ and $\mathbb{E}\left(X_tY_t\right)$ is well known :

- $\mathbb{E}\left(X_t^2\right) = \frac{1-e^{-2\kappa_X T}}{2\kappa_X}$
- $\mathbb{E}\left(Y_t^2\right) = \frac{1-e^{-2\kappa_Y T}}{2\kappa_Y}$
- $\mathbb{E}\left(X_tY_t\right) = \rho_{X,Y}\frac{1-e^{-(\kappa_X+\kappa_Y)T}}{\kappa_X+\kappa_Y}$

As a result, the Bergomi's SLV model is fully defined.

7.1.2 Heston SLV model

This is much more classical:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = \sigma(t, S_t^{s_{ini}})\sqrt{V_t}dW_t^{\mathbb{Q}} \\ dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho dt \end{cases} \tag{44}$$

Which is much easier to simulate.

7.1.3 SABR SLV model

The model is defined as such :

$$\begin{cases} dS_t^{s_{ini}} = \mu_t\sigma(t, S_t^{s_{ini}})\left(S_t^{s_{ini}}\right)^{\beta}dW_t^{\mathbb{Q}} \\ d\mu_t = \alpha\mu_t dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho dt \end{cases} \tag{45}$$

7.2 Hybrid SLV models

We present the Hull&White Bergomi's SLV model. It's the same as (40) but there is an additional stochastic equity drift :

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}})\sqrt{\xi_t^T}dW \\ dr_r = \kappa(\theta(t) - r_t)dt + \xi dB_t^{\mathbb{Q}} \\ \xi_t^T = \xi_0^T f^T(t, x_t^T) \\ f^T(t,x) = exp\left(2\sigma x - 2\sigma^2 h(t,T)\right) \\ x_t^T = \alpha_\theta\left(\left(1-\theta\right)e^{-\kappa_X(T-t)}X_t + \theta e^{-\kappa_Y(T-t)}Y_t\right) \\ \alpha_\theta = \left(\left(1-\theta\right)^2 + \theta^2 + 2\rho_{X,Y}\theta\left(1-\theta\right)\right)^{-\frac{1}{2}} \\ X_t = \int_0^t e^{-\kappa_X(t-s)}dW_s^X \\ Y_t = \int_0^t e^{-\kappa_Y(t-s)}dW_s^Y \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}}\right\rangle_t = \rho dt \end{cases} \tag{46}$$

8 Calibration of SLV models - Numerical simulations

The simulation will be only in the case of the simple (Non hybrid) Bergomi's SLV model (42), as the hybrid case is quite analogous.
This time, rather than considering an example of a deterministic Dupire's local volatility that depend on the rate curve, we will generate a Dupire-like local volatility surface extended from the SSVI parametrisation which is by definition arbitrage free (specially the butterfly arbitrage free if calibrated well).
I will thus begin this section with a short introduction of the SSVI local volatility parametrisation. All the related formulas are obtained by direct computations.

8.1 SSVI local volatility parametrisation

8.1.1 SSVI implied volatility parametrisation

SSVI smile parametrisation is a implied volatility parameter model, to be precised, it is a parametrisation of the total variance. It is defined as the following :

$$w(\theta_t, \kappa) = \frac{\theta_t}{2}\left(1 + \rho\phi(\theta_t)\kappa + \sqrt{\phi^2(\theta_t)\kappa^2 + 1 + 2\rho\phi(\theta_t)\kappa}\right) \tag{47}$$

Where

- θ_t and κ are meant to be respectively the maturity and the log-moneyness
- ϕ and θ_t are two function such that:

$$\lim_{t \longrightarrow 0} \theta_t \phi(\theta_t) \in \mathbb{R} \tag{48}$$

What is practical is that all the quantities of interest (partial strike and maturity derivatives) in order to build the SSVI local volatility using the formula of the appendix are given by closed formulas:

$$\begin{cases} \partial_\kappa w = \frac{\theta_t \phi(\theta_t)}{2}\left(\rho + \frac{\kappa\phi(\theta_t)+\rho}{\sqrt{\phi^2(\theta_t)\kappa^2+1+2\rho\phi(\theta_t)\kappa}}\right) \\ \partial_{\kappa,\kappa} w = \frac{\theta_t \phi^2(\theta_t)}{2} \frac{1-\rho^2}{(\phi^2(\theta_t)\kappa^2+1+2\rho\phi(\theta_t)\kappa)^{\frac{3}{2}}} \\ \partial_t w = \frac{\partial\theta_t}{2\partial t}\left(1 + \frac{\kappa\phi(\theta_t)+\rho}{\sqrt{\phi^2(\theta_t)\kappa^2+1+2\rho\phi(\theta_t)\kappa}} + \kappa\partial_\theta\left(\theta_t\phi(\theta)\right)\left(\left(\frac{\kappa\phi(\theta_t)+\rho}{\sqrt{\phi^2(\theta_t)\kappa^2+1+2\rho\phi(\theta_t)\kappa}}\right) + \rho\right)\right) \end{cases} \tag{49}$$

We denote for the rest :
$z = \sqrt{\phi^2(\theta_t)\kappa^2 + 1 + 2\rho\phi(\theta_t)\kappa}$

8.1.2 SSVI local volatility parametrisation

Using the results from the appendix, we have the local volatility parametrisation surface by a semi-closed formula (depend on ϕ and θ_t derivatives):

$$\sigma_{loc}^2 = \frac{\frac{\partial\theta_t}{\partial t}\left(1 + \frac{\kappa\phi(\theta_t)+\rho}{z} + \kappa\partial_\theta\left(\theta_t\phi(\theta)\right)\left(\left(\frac{\kappa\phi(\theta_t)+\rho}{z}\right) + \rho\right)\right)}{2a - 2b\theta_t\phi(\theta_t) - \frac{c\theta_t^2\phi(\theta_t)^2}{8}} \tag{50}$$

Where :

$$\begin{cases} a = \left(1 - \frac{\kappa\partial_\kappa w}{2w}\right)^2 \\ b = \frac{z^2+(2\rho^2+\rho\phi(\theta_t)\kappa-1)z-2(1-\rho^2)}{2z^3} \\ c = \frac{(\phi(\theta_t)\kappa+\rho+\rho z)^2}{4z^2} \end{cases} \tag{51}$$

So, I used (49) to generate σ_{Dup} that is in the particle method algorithm.

8.2 Calibration results

The Bergomi's stochastic volatility is calibrated separately, we take the following values:

- $\theta = 0.3$
- $\rho_{XY} = 0.3$
- $\kappa_X = 4$
- $\kappa_Y = 0.12$
- $\sigma = 1$
- $\xi_0 = 0.3$

For the SSVI parameters, we consider:

- $\phi_{SSVI}(x) = 0.93x^{-0.45}$
- $\theta_{SSVI}(x) = 0.024x$

Given a set of strikes and a fixed maturity T, we generate σ_{Dup} as a SSVI local volatility :

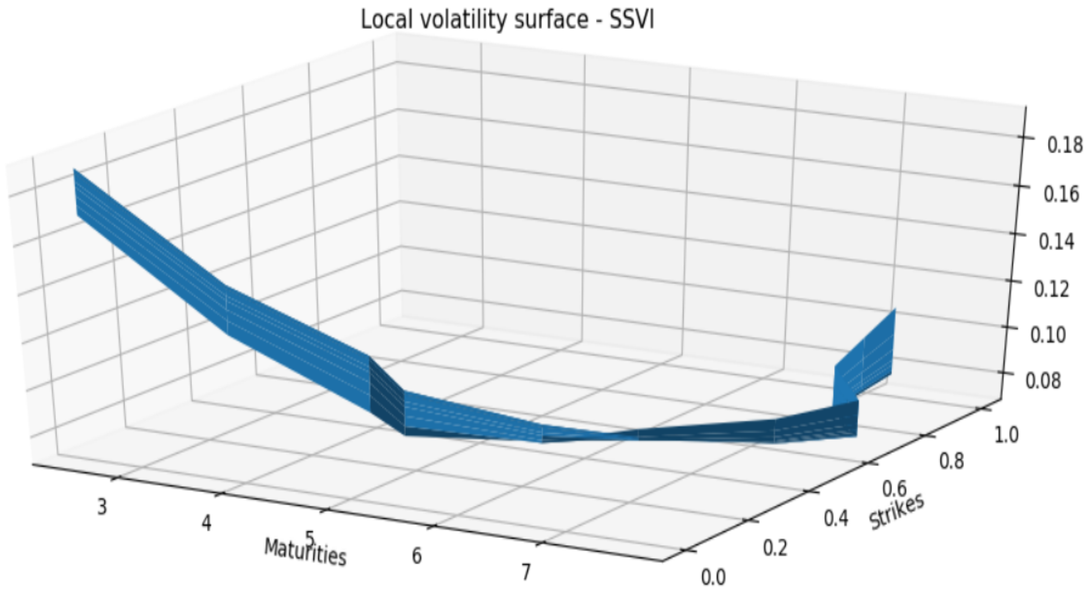

Figure 4: SSVI local volatility surface

We obtain by linear grid interpolation of the local volatility of the SLV model by the particle method algorithm:

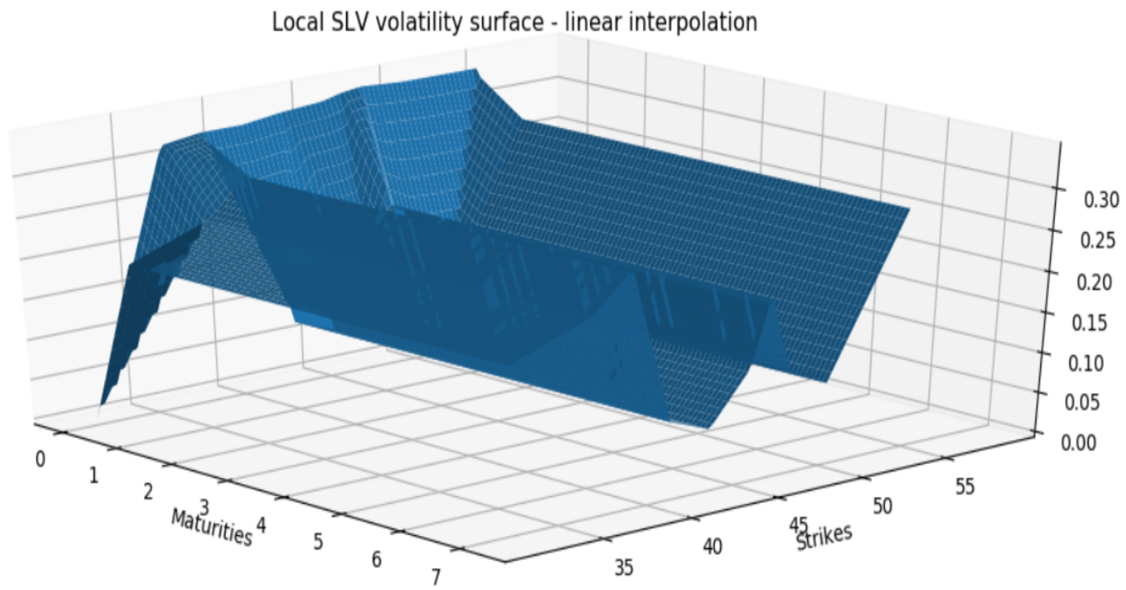

Figure 5: SLV local volatility surface calibrated - 100 particles

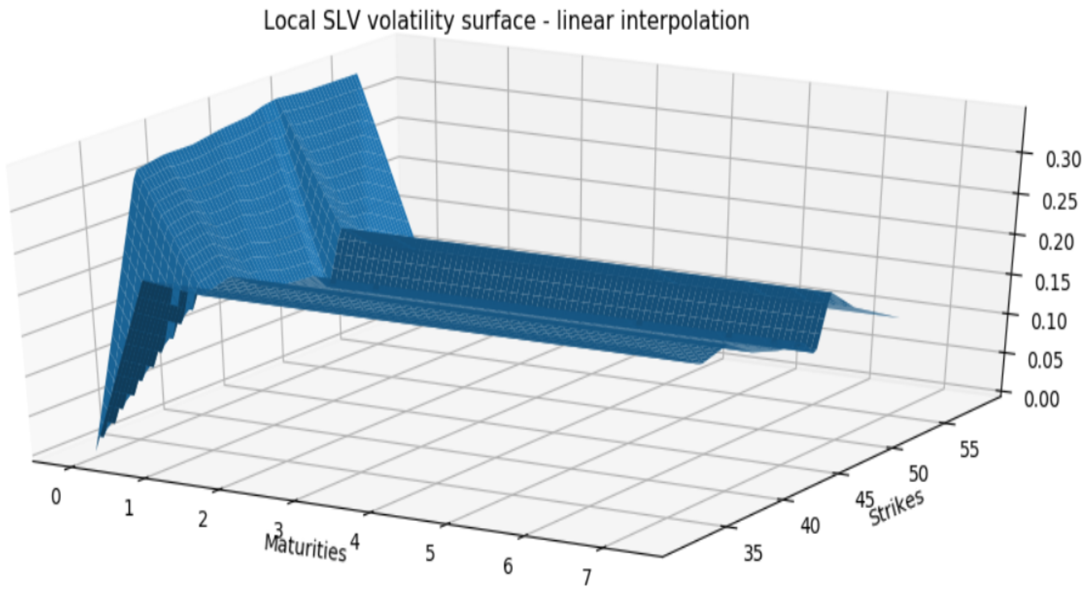

Figure 6: SLV local volatility surface calibrated - $1e + 3$ particles

8.3 Main remarks

- The SSVI local volatility parametrisation is very sensitive to θ and ϕ functions. That's why it is worth being calibrated well.

- All the incertainties related to the strike and maturity interpolation (boundary extrapolation for instance) are encompassed in σ_{Dup} construction. As a result if the construction is robust (by using monotone convex interpolation schemes such as steinman) the calibration is robust.
- The case of the Hybrid SLV is treated with the same methodology
- The linear grid interpolation is done only to be able to plot the calibrated SLV local volatility on a graph.
- According to the figures 5 and 6, we see that by enhencing the particle number the SLV local volatility surface is regularized for extreme strikes.
- The computational time is strongly related to the number of interacting particles. It is advised to consider a considerable ($\approx$ 1e+5) number of particles to obtain a robust calibration.

Appendix : Local volatility and implied volatility

The goal of this appendix is to justify the link between Dupire's local volatility with deterministic rates and the equity's smile (implied volatility).

Dupire's formula for deterministic interest rates

We recall the hybrid local volatility models with stochastic rates:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \tag{52}$$

Where:

$$\sigma^2_{det}(T,K) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \tag{53}$$

In the case of deterministic time dependent rates, r_T goes out of the expectation as we have already said, and we have the Dupire's local volatility for time dependent rates:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) + \frac{K r_T \partial_K C(T,K)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \tag{54}$$

Thus :

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + K r_T \partial_K C(T,K)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \tag{55}$$

We introduce the implied volatility by its basic definition:

$$C(T,K) = C_{BS}(T,K,\sigma_{imp}(T,K)) \tag{56}$$

And we obtain by direct computations the followong property:

Propriété 1 *The local variance $v : (T,K) \longrightarrow \sigma^2(T,K)$ is expressed in function of the total implied variance $w : (T,K) \longrightarrow T\sigma^2_{imp}(T,K)$ by:*

$$v(T,K) = \left(\frac{\partial_T w}{1 + \frac{K}{w}\partial_K w + \frac{1}{2}\partial_{KK} w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{K^2}{w}\right)(\partial_K w)^2}\right)(T,K) \tag{57}$$

Thus, in the hybrid local volatility surface, controling the hybrid local volatility function and fixing the model parameters allowed to se directly its impact on the equity smile's deformation.

Particular test case

From the previous formula, we have:

$$v(T,K) = \left(\frac{\sigma^2_{imp} + 2T\sigma_{imp}\partial_T \sigma_{imp}}{1 + \frac{K}{w}\partial_K w + \frac{1}{2}\partial_{KK} w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{K^2}{w}\right)(\partial_K w)^2}\right)(T,K) \tag{58}$$

If we consider a flat equity smile (a constant implied volatility for all strikes and maturities), we notice that the local variance coincides with the implied variance.

References

[1] G. DOS REIS, S. ENGELHARDT,A. SMITH, *Simulation of McKean-Vlasov SDEs with superlinear growth*