Multiscale Orenstein-Uhlenbeck stochastic volatility model

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The issue of financial data autocorrelation is a crucial when it comes to build efficient and scalable models. In that context, the multi-scaling seems to be the recurrent feature that needs to be considered. Given an unerlying asset, we have two types of non linear structures, respectively ***Volatility autocorrelation*** which is a long memory feature, and ***leverage autocorrelation*** which is a short memory one. In fact, there is numerous stochastic volatility models in the literature and specific scale consideration is not a classical model artifact. However, one alternative to take that into account is to consider the multiscale version of the multidimensional Ornstein-Uhlenbeck with a stochastic mean level.

1 Orenstein-Uhlenbeck volatility process prerequisits

It's well known that tha Black-Scholes model is old fashioned in many underlying markets. One reason in favor of the previous assumption apart from the implied volatility asset smiles is log normal goodness of fit tests of asset return time series (Kolmogorov-Smirnov type for instance) which are generally in favor of a heavy tailed distribution. Thus, the idea to extend this previous financial machinery came with stochastic volatility models.

In the high frequency context, some stylized financial facts must be reflected by the model as mentioned in the paper **"Empirical properties of asset returns : stylized facts and statistical issues"**, for instance :

- Heavy tailed asset distributions
- A correlation between the traded volumes and the volatility (generally negative)
- High return variability
- Leverage effect : the asset returns and their respective volatility are negatively correlated

Here is a visual example with the S&P500 and the VIX indices :

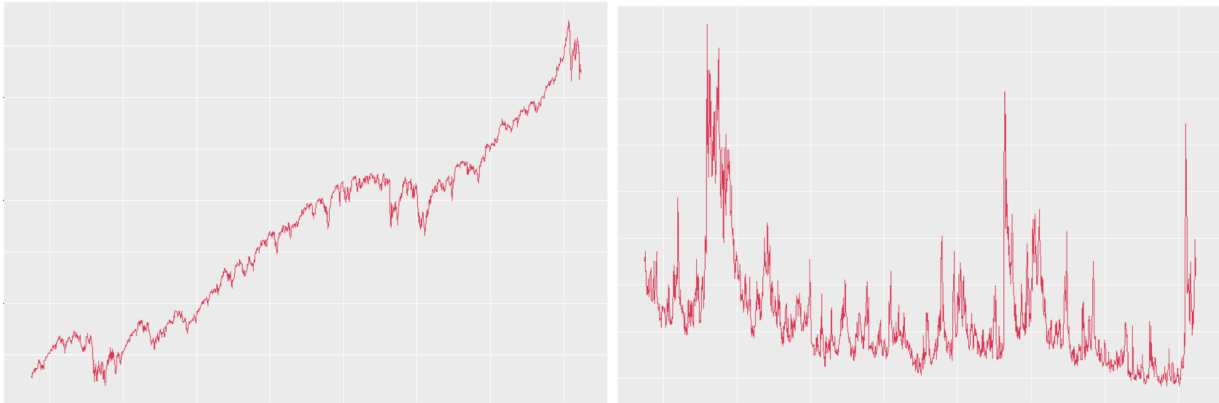


FIGURE 1 – S&P500 returns and VIX variations between 2012 and 2017

In the Heston model :

$$\begin{cases} dS_t = S_t r dt + S_t \sqrt{V_t} dW_t^S \\ dV_t = \kappa \left(\theta - V_t\right) dt + \xi \sqrt{V_t} dW_t^V \\ d\left\langle W_t^S, W_t^V \right\rangle = \rho dt \end{cases} \tag{1}$$

The leverage effect is captured by the correlation parameter ρ and is calibrated as a negative market data parameter. However, the disadvantage of the Heston model as well as all the classical stochastic volatility models (Stein-Stein, Hull& White,...) is that they're mono-scale models.

The aim is to be able to adapt the well known Ornstein-Uhlenbeck stochastic volatility model into a multi-scale one.

2 Extended Ornstein-Uhlenbeck model

In this section, we're going to present the Ornstein-Uhlenbeck stochastic volatility model.

Let's consier the driftless time dependent bachelier asset model :

$$dX_t = \sigma_t dW_t^1 \tag{2}$$

Where W^1 is a standard Brownian motion and σ the stochastic volatility that follows an Ornstein-Uhlenbeck dynamic :

$$d\sigma_t = \alpha(m_t - \sigma_t)dt + kdW_t^2 \tag{3}$$

The mean level m is itself an Ornstein-Uhlenbeck process :

$$dm_t = \alpha_0(m_0 - m_t)dt + k_0 dW_t^3 \tag{4}$$

The leverage effect is taken into account as follow : we consider (W^1, W^2, W^3) the $\mathbb{R}^3$-valued Brownian motion with the covariance matrix :

$$\begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The extended Ornstein-Uhlenbeck is then a multifactorial model :

$$\begin{cases} dX_t = \sigma_t dW_t^1 \\ d\sigma_t = \alpha(m_t - \sigma_t)dt + kdW_t^2 \\ dm_t = \alpha_0(m_0 - m_t)dt + k_0 dW_t^3 \end{cases}$$

2.1 Properties of the extended Ornstein-Uhlenbeck model

According to the formula (6), we claim :

Proposition 1 *The process m est is stationary and gaussian :*

$$m_t = m_0 + k_0 \int_{\infty}^{t} e^{-\alpha_0(t-s)} dW_s^3 \tag{5}$$

Its covariance structure is :

$$k(s,t) = k_0 \int_{\infty}^{inf\{t,s\}} e^{-\alpha_0(t-u)} e^{-\alpha_0(s-u)} du \tag{6}$$

Proposition 2 *The process (5) has a closed form expression :*

$$\sigma(t) = k_0 \int_{\infty}^{t} (k\xi_2(s) + \alpha m(s)) e^{-\alpha(t-s)} ds \tag{7}$$

Where $(\xi_i)_{i\in\{1,2,3\}}$ is a gaussian white noise with the covariance structure :

$$corr_{\xi}(t,s) = \mathbb{E}(\xi_i(t)\xi_j(s)) = \rho_{i,j}\delta(t-s) \tag{8}$$

Proposition 3 *The autocorrelation volatility function is expressed as follows :*

$$corr_{\sigma}(t,s) = m_0^2 \left(1 + \left(\nu^2 - \frac{\lambda\nu_0^2}{1+\lambda^2}\right) e^{-\alpha(|t-s|)} + \frac{\nu_0^2}{1+\lambda^2} e^{-\alpha_0(|t-s|)}\right) \tag{9}$$

Where :
- $\lambda = \frac{\alpha_0}{\alpha}$
- $\nu_0^2 = \frac{k_0^2}{2m_0^2\alpha_0}$
- *We denote* $\hat{\nu}_0^2 = \frac{\nu_0^2}{1+\lambda}$
- $\nu^2 = \frac{k^2}{2m_0^2\alpha}$

Which depends only on the (5)'s parameters

Here are some qualitative remarks :

Remarque 1 *We claim that :*

1. *If the mean level mean reversion is neglected against the volatility mean reversion, the volatility returns to its mean level more frequently than the mean level process. Thus, the volatility tend to be less correlated*
2. *If the volatility of volatility is high, the volatility process tends to be self correlated*
3. *Each the the model factors can be assimilated to a slow and fast factor. That's how the **multiscale** fact is constructed*

3 Leverage effect

The leverage effect is defied as the correlation between the asset returns and the volatility :

$$L(\tau) = \frac{1}{Z}\left\langle dX(t+\tau)^2 dX(t)\right\rangle \tag{10}$$

Where $Z = \left\langle dX(t)^2\right\rangle^2$ is a normalisation constant. According to the extended Ornstein-Uhlenbeck model formulation, we have a leverage effect closed formula.

Proposition 4 *Using the Novikov theorem, the leverage has the following closed formula :*

$$L(\tau) = \frac{2\rho k e^{-\alpha\tau} 1_{\tau>0}\left\langle \sigma(t+\tau)\sigma(t)\right\rangle}{\left\langle \sigma(t)^2\right\rangle^2}, \forall \tau \in \mathbb{R} \tag{11}$$

This formula is obtained with a basic computation of the X's covariance function. Using (10), we obtain the correlation function of σ as well. As a result :

$$L(\tau) = 1_{\tau>0} A(\tau) e^{-\alpha\tau} \tag{12}$$

Where A is the following function :

$$A(\tau) = \tfrac{2\rho\nu\sqrt{2\alpha}}{m_0(1+a+b)^2}\left(1 + ae^{-\alpha\tau} + be^{-\alpha_0\tau}\right)$$

Where :
$a = \nu^2 - \frac{\lambda\nu_0^2}{1-\lambda^2}$
$b = \frac{\nu_0^2}{1-\lambda^2}$

> **Remarque 2** ρ *must be calibrated with the constraint of being bounded between -1 and 0 in order to reflect the market stylised fact.*

4 Extended Ornstein-Uhlenbeck model calibration

Usually, we calibrate the model with liquid market instruments as european calls and puts. For instance, those under Dow-Jones daily index applying th following machinery :

$$\Theta^* = \underset{\Theta\in\mathbb{R}^d}{\arg\min} \sum_{i,j} \left(call_{model}\left(K_i, T_j, \Theta\right) - call_{market}(K_i, T_j)\right)^2 \tag{13}$$

Where $\Theta^* \in \mathbb{R}^d$ is the model parameteric set.

Another way is to exploit the leverage effect and the return variance correlation as we have closed formulas. As a result, the calibrated parameters are those that enables this correlation to fit with the market value.

To perform the clacssical calibration approach, these elements are required :

1. **An extended Ornstein-Uhlenbeck pricer.**
2. **Quoted calls and put curves with sufficiant strikes and maturities.**
3. **As the functional is highly non convex, an regular free optimisation algorithm is needed.**

In the rest, the interest will be focused on the second calibration approach.

4.1 ν, $\hat{\nu}_0$ and m_0 calibration

We denote :

- $(X_t)_t$ the Dow-Jones index return process
- Δt a fixed time shift
- $\Delta X_t = X_{t+\Delta t} - X_t$
- $dX_t = X_{t+dt} - X_t$

We have :

$$Var(\Delta X_t) = m_0^2 \left(1 + \nu^2 + \hat{\nu}_0^2\right) \tag{14}$$

$$Var\left((\Delta X_t)^2\right) = 2m_0^4 \left(4\left(1 + \nu^2 + \hat{\nu}_0^2\right) - 3\right) \Delta t^2 \tag{15}$$

The expressions bellow have a closed formula and a corresponding market value.
As a result :

$$\frac{1}{(1 + \nu^2 + \hat{\nu}_0^2)^2} = \frac{4}{3} - \frac{1}{6} \frac{Var\left((\Delta X_t)^2\right)}{\left(Var\left(\Delta X_t\right)\right)^2} \tag{16}$$

Thus, (17) et (15) enable to calibrate m_0 and $\nu^2 + \hat{\nu}_0^2$.

4.2 Calibration of the other parameters using the return and leverage correlation

4.2.1 Calibration of the return correlation function

We claim that for any shift time τ :

$$Corr\left((dX_{t+\tau})^2, (dX_t)^2\right) = \frac{1}{Var\left(dX_t^2\right) Var\left(dX_{t+\tau}^2\right)} \left\langle \left\langle dX_t^2 dX_{t+\tau}^2 \right\rangle \right\rangle \tag{17}$$

the X dynamic enables us to obtain :

$$Corr\left((dX_{t+\tau})^2, (dX_t)^2\right) = \frac{\left\langle \sigma_t, \sigma_{t+\tau} \right\rangle^2 - \left\langle \sigma_t \right\rangle^4}{4\left\langle \sigma_t^2 \right\rangle^2 - 3\left\langle \sigma_t \right\rangle^4} \tag{18}$$

Thus, we have the closed formula :

$$Corr\left((dX_{t+\tau})^2, (dX_t)^2\right) = N\left(a\left(2 + ae^{-\alpha\tau}\right)ae^{-\alpha\tau} + b\left(2 + ae^{-\alpha_0\tau}\right)ae^{-\alpha_0\tau} + 2abe^{-(\alpha_0+\alpha)\tau}\right) \tag{19}$$

Where :
$N = \frac{1}{1+8\left(\nu^2+\nu_0^2\right)+4\left(\nu^2+\nu_0^2\right)^2}$
assuming $\tau >> \alpha$ (a large shift time), we obtain the following approximation :

$$Corr\left((dX_{t+\tau})^2, (dX_t)^2\right) = Nb\left(2 + be^{-\alpha_0\tau}\right)be^{-\alpha_0\tau} \tag{20}$$

Having m_0 and $\nu^2 + \hat{\nu}_0^2$ calibrated, the N is calibrated as well as all the other parameters.
The leverage identification will enable to fit ρ.

4.2.2 Leverage calibration

We recall the leverage function of the extended Ornstein-Uhlenbeck model :

$$L(\tau) = 1_{\tau>0}A(\tau)e^{-\alpha\tau} \tag{21}$$

$A(\tau) = \frac{2\rho\nu\sqrt{2\alpha}}{m_0(1+a+b)^2}\left(1 + ae^{-\alpha\tau} + be^{-\alpha_0\tau}\right)$
$a = \nu^2 - \frac{\lambda\nu_0^2}{1-\lambda^2}$
$b = \frac{\nu_0^2}{1-\lambda^2}$

The only unknown parameter is this formula is indeed ρ. we will assume a positive shift time τ. Thus :

$$L(\tau) = \frac{2\rho\nu\sqrt{2\alpha}}{m_0(1+a+b)^2}\left(1 + ae^{-\alpha\tau} + be^{-\alpha_0\tau}\right)e^{-\alpha\tau} \tag{22}$$

Then, the right 0 limit of the leverage is obtained :

$$L(0+) = \frac{2\rho\nu\sqrt{2\alpha}}{m_0(1+a+b)} \tag{23}$$

We have now a simple expression of $L(0+)$ that can be fitted with its corresponding market value :

$$L(0+) = \frac{1}{Z}\left\langle dX(t+)^2 dX(t)\right\rangle \tag{24}$$

5 Extensions : Extended Ornstein-Uhlenbeck model validation

We target in this section to validate the model.
We continue with the latter calibrated model with parameters :

Model parameter	Calibrated value
$\nu^2 + \hat{\nu}_0^2$	0.18
m_0	18.9 % year^{-2}
α	0.1 days^{-1}
α_0	1.3 $10^{-3} days^{-1}$
a	0.14
b	0.04
λ	1.3 10^{-2}
k	2 $10^{-3} days^{-1}$
k_0	1.2 $10^{-4} days^{-1}$
ρ	-0.48

5.1 Extended Ornstein-Uhlenbeck pricer

The pricer is developed with the following architecture :

1. **BlackScholesPricerCall : european Black-Scholes call pricer**

2. **HestonPricer : european heston call pricer**

3. **OrnsteinUhlenbeckPricer : extended Ornstein-Uhlenbeck european call pricer**

4. **ImpliedVolSurface : implied volatility surface of the Heston and the extended Ornstein-Uhlenbeck models**

The pricer was developed using the Monte Carlo method with the following trifactorial Euler scheme :

$$\begin{cases} X_{t_{k+1}} = X_{t_k} + \sigma_{t_k}\left(W^1_{t_{k+1}} - W^1_{t_k}\right) \\ \sigma_{t_{k+1}} = \sigma_{t_k} + \alpha\left(m_{t_k} - \sigma_{t_k}\right)\delta + k\left(W^2_{t_{k+1}} - W^2_{t_k}\right) \\ m_{t_{k+1}} = m_{t_k} + \alpha_0\left(m_0 - m_{t_k}\right)\delta + k_0\left(W^3_{t_{k+1}} - W^3_{t_k}\right) \end{cases} \quad (25)$$

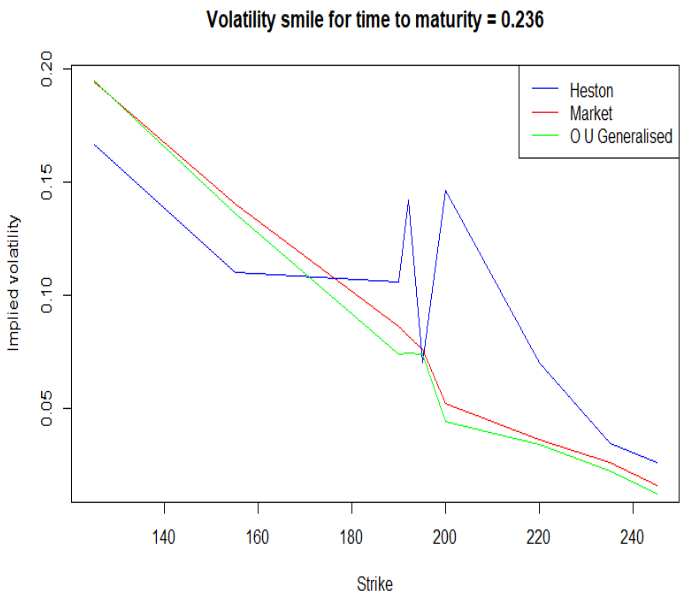
Where $(t_k)_k$ is a uniform subdivision of [0,T] with step δ.

The same for the Heston pricer (rather than the Fast Fourier transformation) :

$$\begin{cases} X_{t_{k+1}} = X_{t_k} + rX_{t_k}\delta + \sigma_{t_k}\left(W^S_{t_{k+1}} - W^S_{t_k}\right) \\ \sigma^2_{t_{k+1}} = \sigma^2_{t_k} + \alpha\left(m_0 - \sigma^2_{t_k}\right)\delta + \xi\left(W^V_{t_{k+1}} - W^V_{t_k}\right) \end{cases} \quad (26)$$

The same calibrated parameters where kept in the Heston model. We took as a risk free rate the LIBOR rate up to the maturity 0.236.

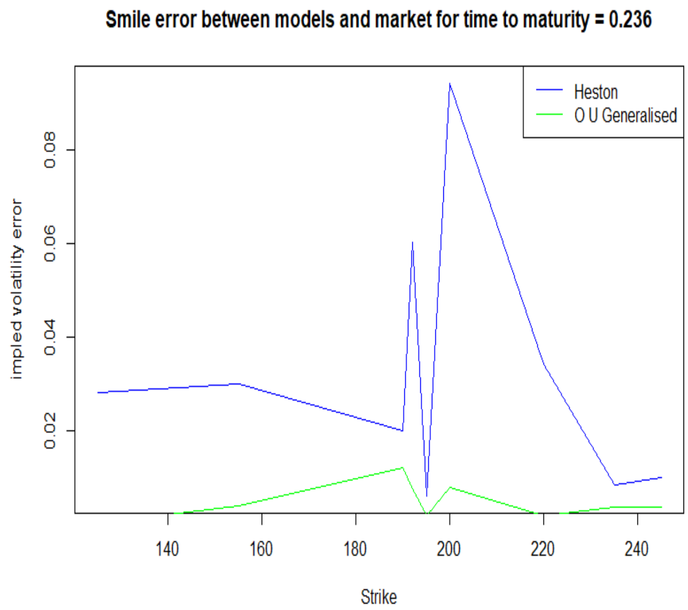
The implied volatility smile up to the maturity 0.236 is shape as follow :



Avec comme smile de marché de smile généré par les calls sur l'indice Dow-Jones daily index.

The smile is far from being regular because only few strikes where considered because of the lack of liquidity of Dow-Jones daily index call options.

The smile error shape is obtained. :



Thus, **The extended Ornstein-Uhlenbeck replicates better the marekt smile than the Heston model**.

In these graphs we lack the multiscale effect.

In the following paper :

A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model, J.-P. Fouque and M. Lorig, SIAM Journal on Financial Mathematics

The multiscale version of Heston was tackled :

$$\begin{cases} dS_t = S_t r dt + S_t E_t dW_t^S \\ E_t = \sqrt{Z_t} f(Y_t) \\ dY_t = \frac{Z_t}{\epsilon}(\theta - Y_t)\,dt + \xi\sqrt{\frac{Z_t}{\epsilon}}dW_t^Y \\ dZ_t = \kappa(\theta - Z_t)\,dt + \sigma\sqrt{Z_t}dW_t^Z \\ d\left\langle W_t^S, W_t^Y\right\rangle = \rho_{s,y}dt \\ d\left\langle W_t^S, W_t^Z\right\rangle = \rho_{s,z}dt \\ d\left\langle W_t^Z, W_t^Y\right\rangle = \rho_{z,y}dt \end{cases} \tag{27}$$

Where ϵ is a fixed parameter.

The calibration was performed using (14) with a minimisation per maturity and per strike of the square error between the model and the market smile.

We obtain the following smiles as functions of the log-moneyness for several maturities (we consider here the market of the S&P500) :

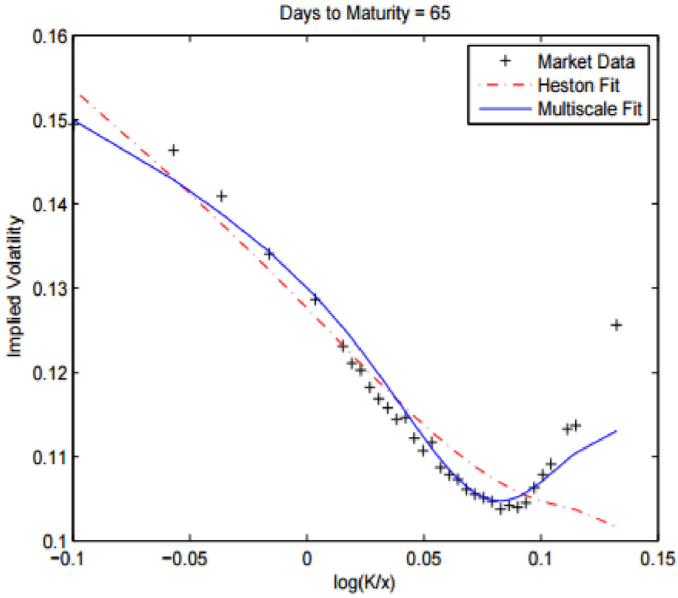


FIGURE 2 – S&P500 Smile - Days to maturity 65

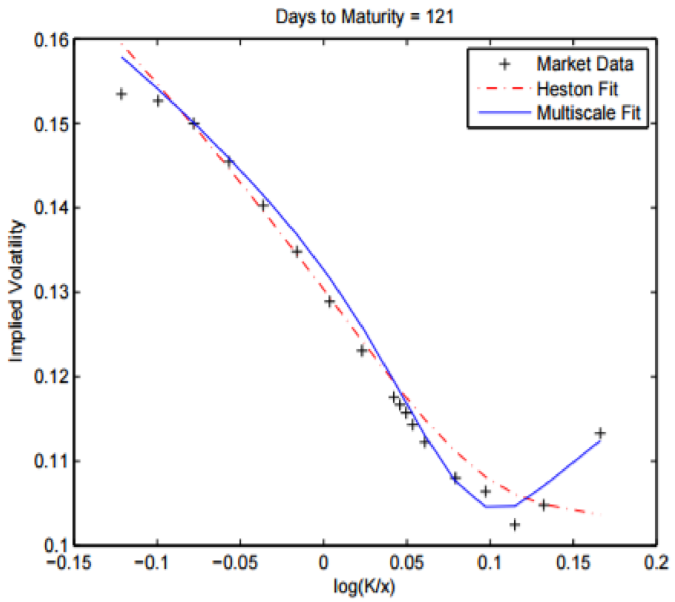


FIGURE 3 – S&P500 Smile - Days to maturity 121

Which reflects that the extended Ornstein-Uhlenbeck has a similar multiscale behaviour.

Bibliography

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- A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model, J.-P. Fouque and M. Lorig, SIAM Journal on Financial Mathematics