Master degree: "Probabilités et finance"

Master thesis

Local volatility models with stochastic interest rates

A thesis submitted to fulfill the recquirements of the Master of science "Probabilités et finance"

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I, Othmane ZARHALI, declare that the present Master thesis and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly during my internship in the quantitative research department at Finastra.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

Signed:
Othmane ZARHALI

"*Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism.*"

Dave Barry

Abstract

The aim of the internship is to study the hybrid equity-rate model with local volatility and stochastic interest rate. It was about the study of the hybrid model for special interest rate models (Gaussian models) in term of pricing validation and Smile dynamic analysis and specially to quantify the impact of the rates's stochasticity in the hybrid equity local volatility surface. Afterwards, a local volatility calibration in the case of stochastic interest rates using two main approaches was performed.

Keywords : Local volatility models, Stochastic interest rates, Hybrid equity-rates model, Stochastic analysis, Malliavin calculus, PDE, Monte Carlo methods, Numerical simulations.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my intern advisor Dr Arnaud RIVOIRA for receiving me as a member of the quantitative research team of FINASTRA, for the continuous support during my work and the multiple propositions he gave me to perform the goals targeted, also for his patience, motivation, and immense knowledge. His guidance helped me. I could not have imagined having a better intern supervisor and mentor for my end of study intern.

Besides my advisor, I would like to thank the rest of quantitative research department team : Dr. Martial MILLET, Mr Benoit GUITARD, Mr Yaya DIA DJIBRIL and Mr Zhang YU for the fruitful lunches that we shared and the prosper working atmosphere.

My sincere thanks also go to the teaching team of the Master "Probabilités et Finance" for the high quality of the academic content of the Master, with a special thought to Mr Gilles PAGES and Nicole EL KAROUI because their respective courses had a paramount importance and constituted for me a considerable academic support during this intern.

Last but not least, I would like to thank from the bottom of my heart all my family members for their inconditional support during this year without whom this fulfillment would have no existence.

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Chapter 1

Introduction and context of the internship

1.1 Brief presentation of Finastra

Finastra is a financial technology company based in London. The firm was constistuted in late 2017 through the combination of D+H (Davis & Henderson) and Misys, after Vista Equity Partners acquired Misys in June 2012 and subsequently purchased D+H in 2017.
Finastra is led by chief executive officer, Simon Paris, who was appointed in June 2018. The company has offices in 42 countries with U.S. 2.1 billion in revenues. The company employs over 10,000 people and has over 9,000 customers across 130 countries.

Main headquarters:

- UK : London
- North American : Toronto, Canada
- USA : New York
- FRANCE : Paris
- ROMANIA : Bucharest
- ASIA : Singapour, Dubai, Hong-Kong, Tokyo

It brings deep expertise and an unrivaled range of pre-integrated solutions spanning retail banking, transaction banking, lending, and treasury and capital markets. With a global footprint and the broadest set of financial software solutions available on the market.

1.2 Custumers of Finastra

In an era of increasing choice and regulation, all customers – corporate, institutional and retail – are demanding greater value from financial services. They expect more agility, innovation, integration and security than ever before.

Finastra delivers flexible and integral software that allows clients to gradually upgrade existing systems through open interfaces.

1.3 Finastra's top management organigram

The top management of Finastra is structured as such:

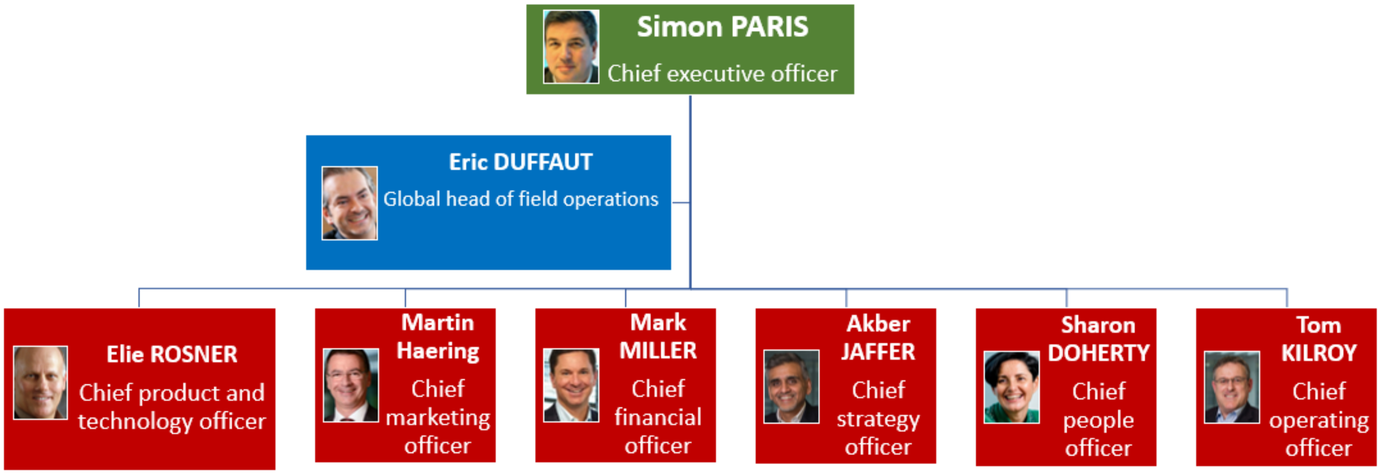

Figure 1.1: Finastra's top management

1.4 The quantitative research department

The Finastra Quantitative Research Team is dedicated to the elaboration and implementation of valuation and risk management models. The financial areas covered are:

- Inflation
- Stocks
- Forex
- Credit
- Commodities

This team is led directly by Dr Martial MILLET, who is also responsible for other quantitative teams in the Paris office. All teams combined, this represents fifteen quants. The fact of having reached this relatively large size has facilitated the introduction of regular presentations on quantitative issues. the goal being to enable everyone to broaden their knowledge and become familiar with different areas, products and models.

During this intern, I was directly affiliated to Dr Arnaud RIVOIRA 's team within the quantitative research department and the intern was supervised by Dr Arnaud RVOIRA himself within Dr Martial MILLET's quantitative research team.

Chapter 2

Preliminaries about local volatility models

The aim of this chapter is to give some basic knowledge about local volatility models that will be useful to understand the main issues tackled during the intern.

As usual we fix the theoretical probability context.

- Let $(\Omega,\mathscr{F},\mathbb{P})$ a complete probability space with a probability measure $\mathbb{P}$ assumed to be the asset historical measure.
- We suppose that a $\mathbb{P}$- classical brownian motion denoted $(B_t)_{t\geq 0}$ is well defined within that space.
- $\mathscr{F}$ is a complete filtration right continuous.
- We denote $\mathbb{Q}$ the risk neutral probability measure.

Unless otherwise stated, T is a fixed maturity.

2.1 Preliminaries about volatility modelling

Let's consider a one dimensional underlying asset with the following risk neutral dynamic:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma_t dW_t^{\mathbb{Q}} \\ S_0^{s_{ini}} = s_{ini} \end{cases} \tag{2.1}$$

We can consider different scenarios:

- **Both r and σ are constant**: This is actually the toy model of Black and Scholes where european call and put options are priced by closed formulas, but the implied volatility surface is flat which is not realistic compared to the market features.
- **r constant and σ deterministic time dependent**: Black and Scholes Closed formulas for european call and put options are also possible with the constant Black and Scholes volatility:

$$\sigma_{BS}^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt \tag{2.2}$$

- **r constant and σ stochastic** : For instance the Stein-Stein (gaussian) or the Heston (mean reverting CIR) models.

  - For affine models, we have a semi-closed european call and put pricing formulas with the inverse Fourier transformation.
  - The heavy tails distributions of assets are captured by the Heston model (the volatility square root effet) and can be refined by some specific modifications of the Heston model (Heston $\frac{3}{2}$, Heston ++,...).
  - The smile is driven by the correlation structure between the underlying and the stochastic volatility, rotation around the money depending on the correlation sign and value (the skew is proportional to the correlation coefficient between the equity and the variance).

- **Both r and σ are stochastic**: For instance an hybrid Heston model with Hull & White spot interest rate. Pricing derivatives are performed via an Hybrid Monte Carlo with approximations on the covariance spot-rate matrix.
  - Based mainly on the computation of $\mathbb{E}^{\mathbb{Q}}\left(\sqrt{v_t}\right)$ by series expansion where $(v_t)_t$ is a CIR process, see [1].

2.2 Local volatility models

Actually, It's well known that the local volatility is so that the risk neutral dynamic of the underlying asset is:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = rdt + \sigma(t, S_t^{s_{ini}})dW_t^{\mathbb{Q}} \\ S_0^{s_{ini}} = s_{ini} \end{cases} \tag{2.3}$$

As a consequence, the volatility is assumed to be time and space dependent. The second case of the previous section is a specific case of local volatility models where the volatility is just time dependent.

2.2.1 Advantages of local volatility models

The main advantages of this modelling choice are:

- The simulation of a unique diffusion to simulate the underlying in a Monte Carlo discretisation scheme, so it's in general less time consuming.
- The perfect calibration to market data (under some regularity conditions on european calls or puts for the Dupire's formula if r is assumed to be constant or time dependent)

The local volatility model is generally not adapted for strong heavy tailed assets, which is a limit in some kind of that model. On way to control the tail of the asset via a local volatility model is to consider **stochastic interest rates**. So the hybrid equity-rate model is formulated:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}})dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \tilde{\sigma}(t, r_t)dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{2.4}$$

In the case of gaussian interest rate models, the risk neutral model turns out to:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}})dW_t^{\mathbb{Q}} \\ r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{2.5}$$

Without any specific assumption on the processes μ and γ a part from the Kunita-Watanabe theorem conditions so that the spot exists.
The aim of the intern is to study this hybrid equity-rate model and to calibrate the local volatility which is a generalisation of the Dupire's approach in the case of constant interest rates.

Chapter 3

Local volatility models with stochastic interest rates

In this chapter, we will tackle the hybrid local volatility calibration using some advanced tools of variational stochastic analysis (Malliavin calculus) and numerical simulations. So this chapter is meant to be more technical and theoretical in order to set up the basis.

3.1 Theoretical context

We will briefly recall the risk neutral hybrid-equity local volatility model in the case of a general stochastic rate model :

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \tilde{\sigma}(t, r_t) dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{3.1}$$

At the end of the journey, the aim of every model is to be used in hedging portfolio derivatives. So this recquires to be able to price derivatives with that model. In our case, the general form of the derivative to be priced is:

$$P_{hyb} = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} h\left(S_T^{s_{ini}}\right)\right) \tag{3.2}$$

Where as usual:

- h the unidimensional payoff
- T a maturity

Obviously, the equity here is supposed to have risk neutral dynamic.
Actually, we can simpify the price by performing a forward neutral change of measure with maturity T:

$$\frac{dQ^T}{dQ}|\mathscr{F}_t = \frac{e^{-\int_0^t r_s ds} B(t,T)}{B(0,T)} \tag{3.3}$$

We obtain:

$$P_{hyb} = B(0,T)\mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left(h\left(S_T^{s_{ini}}\right)\right) \tag{3.4}$$

The price expression (3.4) is:

- Completely model rate independent, as it's just a simple change of measure.

- According to the hybrid model chosen between (2.4) and (2.5) the computation of the zero coupon bond will be achieved by :
  - Closed formula for gaussian rates
  - Monte Carlo for general rates models

Thus, to compute P_{hyb} we have to deduce the forward neutral dynamic of equity anyway.

3.2 Flat local volatility with gaussian interest rates

3.2.1 Forward neutral equity dynamics with gaussian interest rates

We consider gaussian spot rates so as to have some closed formulas and then explain how it works in the case of non gaussian rates. We denote the hybrid equity-spot rate model as in the previous chapter:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{3.5}$$

So the integrated rates are gaussian with mean $\int_0^t \mu_s ds$ and variance $\int_0^t \left(\int_\theta^t \gamma_{\theta,u} du\right)^2 d\theta$. The basic idea is to write under $\mathbb{Q}$:

$$dlog(S_t^{s_{ini}}) = \left(r_t - \frac{\sigma\left(t, S_t^{s_{ini}}\right)^2}{2}\right) dt + \sigma\left(t, S_t^{s_{ini}}\right) dW_t^{\mathbb{Q}} \tag{3.6}$$

So we deduce easily that:

$$S_t^{s_{ini}} = s e^{\int_0^t \left(r_u - \frac{\sigma\left(u, S_u^{s_{ini}}\right)^2}{2}\right) du + \int_0^t \sigma\left(u, S_u^{s_{ini}}\right) dW_u^{\mathbb{Q}}} \tag{3.7}$$

Now we assume the following:

- (H1) **the local volatility is flat** (time dependent).
- (H2) We denote $\Gamma_{\theta,t} = \int_\theta^t \gamma_{\theta,s} ds$ and assume that is time dependent and square integrable.

As a consequence, we have a closed formula of the zero-coupon bond using the famous formula of a Fourier transformation of a gaussian random variable:

$$\forall t \geq 0, B(0,t) = e^{-\int_0^t \mu_s ds + \frac{1}{2}\int_0^t \Gamma_{\theta,t}^2 d\theta} \tag{3.8}$$

By the stochastic Fubini theorem and the change of measure, we end up with :

$$\frac{S_t^{s_{ini}}}{B(t,T)} = \frac{s}{B(0,T)} e^{\int_0^t \Gamma_{\theta,t} dB_\theta^{\mathbb{Q}^{\mathbb{T}}} + \sigma_\theta dW_\theta^{\mathbb{Q}^{\mathbb{T}}} - \frac{1}{2}\int_0^t \sigma_{eq}^2(\theta) d\theta} \tag{3.9}$$

where in t=T:

$$\sigma_{eq}^2(\theta) = \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta \sigma(\theta) \Gamma_{\theta,T} \tag{3.10}$$

Thus, by Lévy's theorem, we obtain:

$$S_T^{s_{ini}} = \frac{s}{B(0,T)} e^{\int_0^T \sigma_{eq}(\theta) d\tilde{W}_\theta^{\mathbb{Q}^{\mathbb{T}}} - \frac{1}{2}\int_0^T \sigma_{eq}^2(\theta) d\theta} \tag{3.11}$$

Where $\tilde{W}^{\mathbb{Q}^{\mathbb{T}}}$ is a $\mathbb{Q}^{\mathbb{T}}$- brownian motion.
So we have the forward neutral dynamics of the asset and the rate by change of measure (Girsanov) which is an exponential martingale under the forward neutral measure as we know.

Hence, in theory, the pricing of a european call and put is exactly as the case of a zero interest rate and a time dependent volatility under $\mathbb{Q}^{\mathbb{T}}$, so we price it with a closed formula with the average volatility (2.2).

But the flat local volatility is still unkown.

3.2.2 Calibration of the flat local volatility surface

Actually, starting from both formulas (3.10) and (3.11) we have two approaches:

Static replication

Actually, the parameter of interest to calibrate in that case is not the flat equity local volatility but it is $\sigma_{eq}(.)$ which depends on the maturity T.

From (3.11) we have:

$$\forall t \geq 0, \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[log\left(\frac{S_T^{s_{ini}}}{\frac{s}{B(0,T)}}\right)\right] = -\frac{1}{2}\int_0^T \sigma_{eq}^2(\theta)d\theta \tag{3.12}$$

In particular for a fixed maturity:

$$\mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[log\left(\frac{S_T^{s_{ini}}}{\frac{s}{B(0,T)}}\right)\right] = -\frac{1}{2}\int_0^T \sigma_{eq}^2(\theta)d\theta \tag{3.13}$$

Then:

$$\frac{\partial \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[log\left(\frac{S_T^{s_{ini}}}{\frac{s}{B(0,T)}}\right)\right]}{\partial T} = -\frac{1}{2}\left(\sigma_{eq}^2(T) + \int_0^T 2\sigma_{eq}(\theta)\frac{\partial \sigma_{eq}(\theta)}{\partial T}d\theta\right) \tag{3.14}$$

As a consequence:

- (3.14) is an ODE in $\sigma(T) = \sigma_{eq}(T)$ that can be solved iteratively and the left hand side can be replicated by Carr-Madan formula.
- for small time to maturities, we can neglect the deterministic integral of the right hand side and we have:

$$\frac{\partial \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[log\left(\frac{S_T^{s_{ini}}}{\frac{s}{B(0,T)}}\right)\right]}{\partial T} \approx -\frac{1}{2}\sigma_{eq}^2(T) \tag{3.15}$$

  Then, we obtain a flat local volatility calibrated to market data.

This is obtained with any gaussian interest rate model.

Non linear ODE approach

This is made via (3.10):

$$\sigma_{eq}^2(\theta) = \sigma^2(\theta) + \Gamma_{\theta,t}^2 + 2\rho_\theta\sigma(\theta)\Gamma_{\theta,t}$$

So the theoretical implied volatility is:

$$\sigma_{th}^2(T) = \frac{1}{T}\int_0^T \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta\sigma(\theta)\Gamma_{\theta,T}d\theta \tag{3.16}$$

It's similar to consider the implied theoretical variance:

$$V_{th}(T) = \int_0^T \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta\sigma(\theta)\Gamma_{\theta,T}d\theta \tag{3.17}$$

In the specific case of Hull & White model with constant equity-rate correlation coefficient:

$$dr_t = (\theta - \alpha r_t)dt + \gamma dB_t^{\mathbb{Q}} \tag{3.18}$$

Which will be used as a reference in the intern. We obtain:

$$\sigma^2(T) = \frac{dV_{th}(T)}{dT} - \Gamma_{0,T}^2 - 2\rho\gamma\int_0^T e^{-\alpha(T-u)}\sigma(u)du \tag{3.19}$$

Which is a non linear ODE that can be resolved by:

- Identification of the term $\frac{dV_{th}(T)}{dT}$ with market data
- Choosing our favorite numerical scheme (finite differences for instance).

Remark 1 *We can simplify the previous ODE by considering a resolution for large time to maturities. We have in the case of H&W:*

$$\sigma_\infty = \sqrt{\frac{dV_{th}(T)}{dT}|T\infty - (1-\rho^2)\left(\frac{\gamma}{\alpha}\right)^2} - \rho\frac{\gamma}{\alpha} \tag{3.20}$$

We denote: $\sigma_\infty^{det^2} = \frac{dV_{th}(T)}{dT}|T\infty$ we then obtain the formula:

$$\sigma_\infty = \sqrt{\sigma_\infty^{det^2} - (1-\rho^2)\left(\frac{\gamma}{\alpha}\right)^2} - \rho\frac{\gamma}{\alpha} \tag{3.21}$$

The resolution of the previous ODE by finite differences for the given constant model parameters:

- $\frac{1}{\alpha} = 10Y$
- $\gamma = 0.5\%$
- $\rho = -0.5$
- $\sigma^{det}(Y) = 0.6$

is:

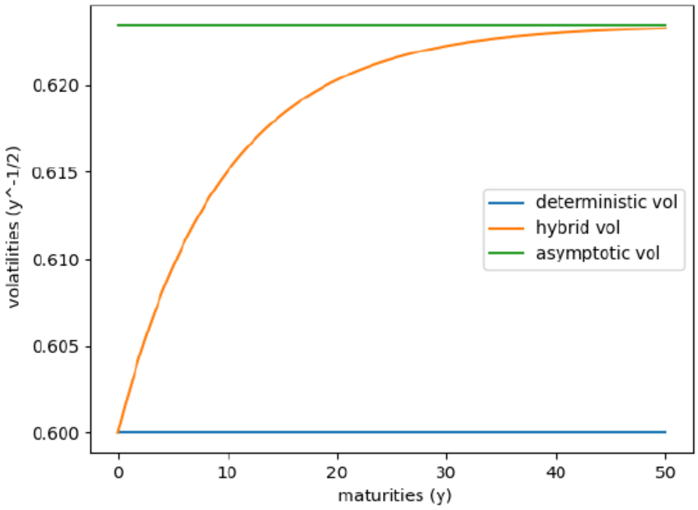

Figure 3.1: ODE calibration for flat local volatility

We actually see that the numerical solution converges to its terminal value for large times to maturity.

Another calibration view point is to see the equation (3.12) is a static replication formula of the log-contract, whose expression at maturity is: $log\left(\frac{S_T^{s_{ini}}}{\frac{s}{B(0,T)}}\right)$, and is related to the forward variance stochastic volatility models. One way to perform the identification of the log-contract is to use the **Carr-Madan** static replication formula. But from the outset, we observe a number of pros and cons of these two previous calibration approaches in the case of an hybrid flat local volatility with Gaussian stochastic interest rates, which will be detailed deeply in the next subsection.

3.2.3 Pros and Cons of the previous calibration approaches

The previous technics are available for the toy local volatility model with flat local volatility surface and gaussian interest rates. But still, this is a way to deduce some qualitative remarks about the previous approach:

Static replication	Non linear ODE method
- Allow exact calibration by Carr-Madan formula	- Easy numerical resolution
- Significant simplification for short maturities	- Significant simplification in the steady state
- Depend on the numerical integration of the Carr-Madan expansion	- Assumes that we have a continuum of implied volatilities or variances for all maturities otherwise we have to bear the interpolation error

Table 3.1: Pros & Cons calibration flat local volatility - H&W rates

3.2.4 Qualitative remarks

- The main disadvantage of the the static replication calibration is that we can't see clearly the effect of the interest rate stochasticity whereas in the non linear ODE approach even in the steady regime we can quantify the adjustment between the hybrid flat local volatility and the deterministic local volatility which depends on the rates's volatility in the H&W model, the speed of mean reversion and the correlation factor.

- Despite the fact that the separation between the Dupire's deterministic rate local volatility and the hybrid adjustment term is not clear, but at least we can control efficiently the adjustment in the steady state by the determination of the volatility rates, the speed of mean reversion and the correlation factor. This is not possible in the static replication approach.

- This adjustment also increases with the maturity as we assume that an arbitrage free implied volatility surface is non decreasing as a function of maturity but not more increasing than a classical square root function of the maturity.

- As the adjustment term depends on the rate model volatility and the correlation factor so it's model dependent and we can not hedge away the correlation risk but at least estimate the amount of cash to live with it..

- The adjustment is the element of interest here because by computing it we can easily switch between the hybrid local volatility that encompasses the rate stochasticity and the deterministic local volatility with deterministic interest rates. This will be more explicit in the following chapters.

3.3 Flat local volatility with general interest rate model

Like before, the model is defined in the following:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t) dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \tilde{\sigma}(t, r_t) dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{3.22}$$

In this case, it's well known that the forward is a $\mathbb{Q}^{\mathbb{T}}$-local martingale. But we do not necessarily have the explicit analytical formula of the rates's volatility in order to compute exactly the forward neutral volatility of the forward neutral equity's dynamic.
Hence, the computations (3.6) and (3.7) will be possible but we can't push the computation further in order to get the exact analytical forward neutral dynamic of the equity.
As a result, it's not trivial to extract the effect of the rates's stochasticity in the flat local volatility model. The only way to calibrate this hybrid model is to try a forward calibration via the identification of the terms of the associated bidimensionnal Fokker-Planck PDE with market data.

3.4 Non flat local volatility model with general interest rate

Let's come back to the general model:

$$\begin{cases} \frac{dS_t^{s_{ini}}}{S_t^{s_{ini}}} = r_t dt + \sigma(t, S_t^{s_{ini}}) dW_t^{\mathbb{Q}} \\ dr_t = b(t, r_t)dt + \sigma(t, r_t) dB_t^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{3.23}$$

We will begin with a basic analoguous computation of the local volatility starting from call or put option prices quoted in the market supposed to be enough time and space regular.

3.4.1 Analytical formula of the local volatility surface

We denote:

$$C(T,K) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\left(S_T^{s_{ini}} - K\right)_+\right) \tag{3.24}$$

By Itô-tanaka formula:

$e^{-\int_0^T r_s ds}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K} dS_t^{s_{ini}} - \int_0^T r_t e^{-\int_0^t r_s ds}\left(S_t^{s_{ini}} - K\right)_+ dt + \frac{1}{2}\int_0^T e^{-\int_0^t r_s ds} dL_t^K$

Where $(L_t^K)_{t\geq 0}$ is the local time of the equity.

Then:

$e^{-\int_0^T r_s ds}\left(S_T^{s_{ini}} - K\right)_+ = (s-K)_+ + \int_0^T e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K} dS_t^{s_{ini}} - \int_0^T r_t e^{-\int_0^t r_s ds}\left(S_t^{s_{ini}} - K\right)_+ dt +$

$\frac{1}{2}K^2\int_0^T e^{-\int_0^t r_s ds}\sigma^2(t,K)\delta_K(S_t)dt$

We assume that the stochastic integral in the asset's dynamic vanishes under expectation, then by Fubini theorem we obtain:

$$C(T,K) = C(0,K) + \int_0^T K\mathbb{E}^{\mathbb{Q}}\left(r_t e^{-\int_0^t r_s ds} 1_{S_t^{s_{ini}}>K}\right) dt + \frac{1}{2}K^2\int_0^T \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^t r_s ds}\sigma^2(t,K)\delta_K(S_t^{s_{ini}})\right) dt$$

By differentiation with respect to maturity, we get:

$$\partial_T C(T,K) = K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) + \frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)\sigma^2(T,K)$$

We conclude then that the local volatility can be expressed as follow:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) - K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{3.25}$$

- The term $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)$ is nothing else but $\frac{\partial^2 C(T,K)}{\partial K^2}$
- All the terms are perfectly calibrable
- The only term that distrubs is $\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)$ which needs a special treatment by involving the forward neutral dynamic of the hybrid local volatility model (no longer a capitalised Doleans-Dade martingale)

Once again we obtain the hybrid local volatility model in the form:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} - \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{3.26}$$

Where the adjustment term that is due to the rates's stochasticity is:

$$Adj(T) = \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{3.27}$$

It depends on:

- The joint distribution of the equity and rate : this encompasses the flat local volatility with gaussian rates where the dependence is related to the rate model parameters and also to the correlation factor
- The probability measure : as we have the choice to compute that term in the risk neutral or the forward neutral probability measure

It may be of interest to recall the analogy between the Dupire's local volatility calibration and the hybrid local volatility via the following remark.

Remark 2 *Actually, according to (3.27) we notice that:*

- *If the interest rates are constant, the hybrid local volatility is nothing but the Dupire's one as* $\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) = -r_T \frac{\partial C(T,K)}{\partial K}$ *and Dupire's local volatility calibrates exactly the call surface as it's just an integrated version of the forward Fokker-Planck PDE.*
- *If rates are time dependent, we can still get out* r_T *out of the expectation and obtain a Dupire like formula with time dependent interest rates.*
- *In the case that we are interested in, which is to consider the interest rates stochastics, there is the adjustment term that seems to be new. In addition to that,* $(T,K) \longrightarrow \sigma_{det}(T,K)$ *is exactly the Dupire's local volatility surface with vanishing interest rate and dividends. As we know how to fit this deterministic local volatility to market call prices (we will detail the approach later) one way is to know how to evaluate the adjustment term to calibrate the hybrid local volatility or, in contrary, given an hybrid local volatility surface, to deduce the adjustment term. All that, with as little as computational efforts as possible.*

The term $\frac{\partial^2 C(T,K)}{\partial K^2}$ is proportional to the forward neutral density of the equity and can be directly calibrated from market call prices.
Thus, the hybrid term $\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)$ will be handled using two approaches:

- **A generic approach** : Brute force Monte Carlo computation under the risk neutral probability space with some boosting schemes (Richardson Romberg extrapolation for the Monte Carlo accuracy).
- **A specific approach in the case of gaussian rates** : "PDE expansion" obtained by Malliavin integration by parts formula.

3.4.2 The Monte Carlo approach

The Monte Carlo, as it's well known, consists of performing a numerical Euler scheme on the model in order to compute a sample of realisations of $r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}$ by the weak approximation:

$$\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) \approx \frac{1}{M}\sum_{i=1}^{M}\left(r_T^{(i)} e^{-\delta \sum_{k=1}^{N} r_{t_k}^{(i)}} 1_{S_T^{(i)}>K}\right) \tag{3.28}$$

Where δ is the discretisation time step , M the length of the Monte Carlo sample and N the length of the time discretisation of $[0,T]$.
Indeed, the Euler scheme of the model is the following:

$$\begin{cases} S_{t_{k+1}}^{s_{ini}} = S_{t_k}^{s_{ini}}(1 + r_{t_k}\delta) + S_{t_k}^{s_{ini}}\sigma(t_k, S_{t_k}^{s_{ini}})\sqrt{\delta}G \\ r_{t_{k+1}} = r_{t_k} + b(t_k, r_{t_k})\delta + \tilde{\sigma}(t_k, r_{t_k})\sqrt{\delta}\left(\rho G + \sqrt{1-\rho^2}\tilde{G}\right) \end{cases} \tag{3.29}$$

Where G and $\tilde{G}$ are two independant Gaussian random variables centred with variance equal to 1. The realisations of the spot rate along the time discretisation are stored in order to simulate the Riemann stochastic integral $\int_0^T r_s ds$ with $\delta \sum_{k=1}^{N} r_{t_k}$.
As a result, we obtain the computation of the hybrid expectation term.

(*P.S : the simulation is done under the risk neutral space and the model is a risk neutral one*)

As we know about Monte Carlo time computation on option pricing, when we have the choice to price an option between a Monte Carlo method and a similar partial differential equation one, we prefer to chose the PDE specially in low dimensions. This is exactly the same in the calibration of the hybrid local volatility surface.
The object of the following subsection is to expose an analytical expansion ("PDE expansion") of the hybrid local volatility.

3.4.3 Iterative PDE in the case of Gaussian rates

As stated before, this formulation is specific to the case of Gaussian rate dynamics. In this section and also in the intern we considered the Hull & White gaussian dynamic for the spot rates as well:

$$dr_t = (\theta - \alpha r_t)dt + \gamma dB_t^{\mathbb{Q}} \tag{3.30}$$

For theoretical reasons, we denote the explicit spot rate as previously but in the forward neutral measure :

$$r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}^{\mathbb{T}}} \tag{3.31}$$

The aim of this part is to find out the aimed analytical extension of the hybrid local volatility. Two approaches were explored, a proof under the forward neutral measure developed jointly between Dr RIVOIRA and myself, and a proof under the risk neutral measure performed entirely by myself written in the appendix.

We introduce the following notations:

$$\Gamma_{t,T} = \int_t^T \gamma_{t,u} du \tag{3.32}$$

$$X_t = \ln\left(\frac{S_t D_{0,t}}{S_0}\right) \tag{3.33}$$

$$r_{0,t} = -\frac{\partial \ln(D_{0,t})}{\partial t} \tag{3.34}$$

$$X_{0,t} = -\ln(D_{0,t}) \tag{3.35}$$

$$r_t = \mu_t^* + \int_0^t \gamma_{s,t} dB_s \tag{3.36}$$

$$\mu_t^* = r_{0,t} + \int_0^t \gamma_{s,t}\Gamma_{s,t} ds \tag{3.37}$$

$$X_t = X_{0,t}^* + \int_0^t (\sigma_X(X_s, s)dW_s + \Gamma_{s,t}dB_s) \tag{3.38}$$

$$X_{0,t}^* = X_{0,t} + \frac{1}{2}\left(\int_0^t \left(\Gamma_{s,t}^2 - \sigma_X^2(X_s, s)\right) ds\right) \tag{3.39}$$

$$dX_t = m_t^* dt + \sigma_X(X_t, t)dW_t \tag{3.40}$$

$$m_t^* = r_{0,t} + \int_0^t \gamma_{s,t} (dB_s + \Gamma_{s,t}) ds - \frac{1}{2}\sigma_X^2(X_t, t) \tag{3.41}$$

$$\frac{dS_t}{S_t} = r_t + \sigma_S(S_t, t)dW_t \tag{3.42}$$

$$r_t = \mu_t^* + \int_0^t \gamma_{s,t} dB_s \tag{3.43}$$

$$\mu_t^* = r_{0,t} + \int_0^t \gamma_{s,t}\Gamma_{s,t} ds \tag{3.44}$$

$$X_t = X_{0,t}^* + \int_0^t (\sigma_X(X_s, s)dW_s + \Gamma_{s,t}dB_s) \tag{3.45}$$

$$X_{0,t}^* = X_{0,t} + \frac{1}{2}\left(\int_0^t \left(\Gamma_{s,t}^2 - \sigma_X^2(X_s, s)\right) ds\right) \tag{3.46}$$

$$dX_t = m_t^* dt + \sigma_X(X_t, t)dW_t \tag{3.47}$$

$$m_t^* = r_{0,t} + \int_0^t \gamma_{s,t} (dB_s + \Gamma_{s,t}) ds - \frac{1}{2}\sigma_X^2(X_t, t) \tag{3.48}$$

Now let's move to the forward neutral with maturity T dynamics by applying Girsanov theorem: The Brownian motions $B_t^{\mathbb{Q}}$ and $W_t^{\mathbb{Q}}$ must be changed the following way to move from the risk neutral probability measure to that of the T forward:

$$dB_s^{\mathbb{Q}} \mapsto dB_t^{\mathbb{Q}^T} - \Gamma_{t,T} dt \tag{3.49}$$

$$dW_s^{\mathbb{Q}} \mapsto dW_t^{\mathbb{Q}^T} - \rho\Gamma_{t,T} dt \tag{3.50}$$

That leads to:

$$r_t = \mu_t^T + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}^T} \tag{3.51}$$

$$\mu_t^T = r_{0,t} + \int_0^t \gamma_{s,t} (\Gamma_{s,t} - \Gamma_{s,T}) ds \tag{3.52}$$

$$X_t = X_{0,t}^T + \int_0^t \left(\sigma_X(X_s, s)dW_s^{\mathbb{Q}^T} + \Gamma_{s,t}dB_s^{\mathbb{Q}^T}\right) \tag{3.53}$$

$$X_{0,t}^T = \frac{1}{2}\left(\int_0^t \left((\Gamma_{s,t} - \Gamma_{s,T})^2 ds - {\sigma_X^T}^2(X_s, s)\right) ds\right) \tag{3.54}$$

$${\sigma_X^T}^2(X_s, s) = \sigma_X^2(X_s, s) + 2\rho\sigma_X(X_s, s)\Gamma_{s,T} + \Gamma_{s,T}^2 \tag{3.55}$$

$$dX_t = m_t^T dt + \sigma_X(X_t, t)dW_t^{\mathbb{Q}^T} \tag{3.56}$$

$$m_t^T = m_t^* - \int_0^t \gamma_{s,t}\Gamma_{s,t} ds - \rho\Gamma_{t,T}\sigma_X(X_t, t) \tag{3.57}$$

Exponential weighted average of vanishing functions property - Amnesic approximation

If $f(0) = 0$ then the exponential decay kills everything except the linear profile in the vicinity of $t = 0$ which reads:

$$\int_0^T f(t)e^{-t}dt \sim f'(0)\left(1-(1+T)e^{-T}\right) \tag{3.58}$$

Noting $\sigma_{\text{det}}(x, T)$ the local volatility function computed from the Dupire's formula i.e. replacing the deterministic rate with $r_{0,T}$ and performing a Malliavin integration by parts, we get:

$$\sigma^2_{\text{det}}(x,T) - \sigma^2(x,T) \triangleq \tilde{v}(x,T) = 2\int_0^T \gamma_{t,T}\frac{\mathbb{E}_T\left\{\frac{\partial H_x(X_T)}{\partial dB_t^{\mathbb{Q}^T}}\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}}dt \tag{3.59}$$

where:

- $\frac{\partial}{\partial dB_t^{\mathbb{Q}^T}}$ is the Malliavin derivative operator with respect to the forward neutral Brownian motion $B^{\mathbb{Q}^T}$.
- δ is the Dirac distribution.
- H is the Heaviside function in x defined by

$$H_x(y) = \begin{cases} 1 & \text{if } y > x \\ 0 & \text{otherwise} \end{cases} \tag{3.60}$$

And from Eq.3.53:

$$\frac{\partial X_T}{\partial dB_t^{\mathbb{Q}^T}} = \Gamma_{t,T} + \rho\sigma(X_t,t) + \int_t^T \left(dW_u^{\mathbb{Q}^T} - \sigma(X_u,u)du\right)\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}} - \rho\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}}\Gamma_{u,T}du - \rho\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}}\Gamma_{u,T}du \tag{3.61}$$

$$= \Gamma_{t,T} + \rho\sigma(X_T,T) + \int_t^T d\varepsilon_u^t \tag{3.62}$$

with:

$$d\varepsilon_u^t = \left(dW_u^{\mathbb{Q}^T} - \sigma(X_u,u)du\right)\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}} - \rho d\sigma(X_u,u) - \rho\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}}du - \rho\frac{\partial\sigma(X_u,u)}{\partial dB_t^{\mathbb{Q}^T}}\Gamma_{u,T}du \tag{3.63}$$

Then:

$$\sigma^2_{\text{det}}(x,T) = (\sigma(x,T) + \rho G_T)^2 + 2\int_0^T \gamma_{t,T}\Gamma_{t,T}dt - \rho^2 G_T^2 + w(x,T) \tag{3.64}$$

where

$$G_T = \int_0^T \gamma_{t,T}dt \tag{3.65}$$

$$w(x,T) = 2\int_0^T \gamma_{t,T}\frac{\mathbb{E}_T\left\{\delta_x(X_T)\int_t^T d\varepsilon_u^t\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}}dt \tag{3.66}$$

Estimation of $w(x,T)$

Let us consider:

$$\phi(t) = \frac{\mathbb{E}_T\left\{\delta_x(X_T)\int_t^T d\varepsilon_u^t\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.67}$$

We have

$$w(x,T) = 2\int_0^T \gamma_{t,T}\phi(t)dt = 2\gamma\int_0^T e^{-t/\tau}\phi(t)dt \tag{3.68}$$

$$= 2\gamma\tau\int_0^{T/\tau} e^{-u}\phi(T-\tau u)du \tag{3.69}$$

Considering $\psi(u) = \phi(T-\tau u)$ and applying the amnesic approximation , we get:

$$\int_0^{T/\tau} e^{-u}\psi(u)du \sim \psi'(0)\left(1-(1+T/\tau)e^{-T/\tau}\right) \tag{3.70}$$

And $\psi'(u) = -\tau\phi'(T-\tau u)$ then $\psi'(0) = -\tau\phi'(T)$. Besides, writing the derivation rule to integral expressions for $\theta < T$:

$$\phi'(\theta) = \frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_\theta^T \frac{\partial d\varepsilon_u^\theta}{\partial t} - \left.\frac{d\varepsilon_u^\theta}{du}\right|_{u=\theta}\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.71}$$

Leading therefore to:

$$\phi'(T) = -\frac{\mathbb{E}_T\left\{\delta_x(X_T)\left.\frac{d\varepsilon_u^{T_-}}{du}\right|_{u=T_-}\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.72}$$

Hence:

$$w(x,T) \sim 2\gamma\tau^2\left(1-(1+T/\tau)e^{-T/\tau}\right)\frac{\mathbb{E}_T\left\{\delta_x(X_T)\left.\frac{d\varepsilon_u^{T_-}}{du}\right|_{u=T_-}\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.73}$$

We then need to evaluate $\left.\frac{d\varepsilon_u^{T_-}}{du}\right|_{u=T_-}$. For that, let us Eq.3.63 for $u = T_-$

$$d\varepsilon_{T_-}^T = \left(dW_{T_-}^{\mathbb{Q}^T} - \sigma(X_{T_-},T_-)dT\right)\frac{\partial\sigma(X_T,T)}{\partial dB_{T_-}^{\mathbb{Q}^T}} - \rho d\sigma_X(X_{T_-},T_-) \tag{3.74}$$

From the chain rule:

$$\frac{\partial\sigma(X_T,T)}{\partial dB_{T_-}^{\mathbb{Q}^T}} = \frac{\partial\sigma(X_T,T)}{\partial x}\frac{\partial X_T}{\partial dB_{T_-}^{\mathbb{Q}^T}} \tag{3.75}$$

And from Eq.3.61:

$$\frac{\partial\sigma(X_T,T)}{\partial dB_{T_-}^{\mathbb{Q}^T}} = \rho\frac{\partial\sigma(X_T,T)}{\partial x}\sigma_X(X_T,T) \tag{3.76}$$

And from Itô's lemma:

$$d\sigma_X(X_T,T) = \Sigma_T dT + \frac{\partial \sigma_X(X_T,T)}{\partial x}\sigma_X(X_T,T)dW_T^{\mathbb{Q}^T} \tag{3.77}$$

with

$$\Sigma_T = \frac{\partial \sigma_X(X_T,T)}{\partial t} + \frac{1}{2}\frac{d\langle X\rangle_T}{dT}\frac{\partial^2\sigma_X(X_T,T)}{\partial x^2} + m_T^*\frac{\partial \sigma_X(X_T,T)}{\partial x} \tag{3.78}$$

By direct computations:

$$\langle X\rangle_T = \int_0^T \left((\sigma_X(X_t,t))^2\right)dt \tag{3.79}$$

And by direct differentiation:

$$\frac{d\langle X\rangle_T}{dT} = \sigma_X^2(X_T,T) \tag{3.80}$$

As a result:

$$d\sigma_X(X_T,T) = \frac{\partial \sigma_X(X_T,T)}{\partial t}dT + \frac{1}{2}\sigma_X^2(X_T,T)\frac{\partial^2\sigma_X(X_T,T)}{\partial x^2}dT + \tag{3.81}$$

$$m_T^T\frac{\partial \sigma_X(X_T,T)}{\partial x}dT + \frac{\partial \sigma_X(X_T,T)}{\partial x}\sigma_X(X_T,T)dW_T^{\mathbb{Q}^T} \tag{3.82}$$

By injecting the previous expression, we obtain:

$$\begin{aligned} d\varepsilon_{T_-}^T = \left(dW_{T_-}^{\mathbb{Q}^T} - \sigma(X_{T_-},T_-)dT\right)\frac{\partial\sigma(X_T,T)}{\partial dB_{T_-}^{\mathbb{Q}^T}} - \rho\frac{\partial \sigma_X(X_T,T)}{\partial t}dT \\ -\rho\frac{1}{2}\sigma_X^2(X_T,T)\frac{\partial^2\sigma_X(X_T,T)}{\partial x^2}dT \\ -\rho(m_T^T)\frac{\partial \sigma_X(X_T,T)}{\partial x}dT - \rho\frac{\partial \sigma_X(X_T,T)}{\partial x}\sigma_X(X_T,T)dW_T^{\mathbb{Q}^T} - \rho r_{0,T}\frac{\partial \sigma_X(X_T,T)}{\partial x}dT \end{aligned} \tag{3.83}$$

By simplifications:

$$\begin{aligned} d\varepsilon_{T_-}^T = -\rho(\frac{\sigma_X(X_T,T)^2}{2})\frac{\partial \sigma_X(X_T,T)}{\partial X}dT - \rho\frac{\partial \sigma_X(X_T,T)}{\partial t}dT - \rho\frac{1}{2}\sigma_X^2(X_T,T)\frac{\partial^2\sigma_X(X_T,T)}{\partial x^2}dT \\ -\rho\left(\left(\int_0^T \gamma_{s,T}dB_s\right)\right)\frac{\partial \sigma_X(X_T,T)}{\partial x}dT - \rho r_{0,T}\frac{\partial \sigma_X(X_T,T)}{\partial x}dT \end{aligned} \tag{3.84}$$

Which leads to:

$$\begin{aligned} \frac{d\varepsilon_{T_-}^T}{dT} = -\rho\left(r_{0,T}\frac{\partial \sigma_X(X_T,T)}{\partial x} + \frac{\partial \sigma_X(X_T,T)}{\partial t} + \frac{\sigma_X(X_T,T)^2}{2}\left(\frac{\partial \sigma_X(X_T,T)}{\partial x} + \frac{\partial^2\sigma_X(X_T,T)}{\partial x^2}\right)\right) \\ -\rho\left(\left(\int_0^T \gamma_{s,T}dB_s\right)\right)\frac{\partial \sigma_X(X_T,T)}{\partial x} \end{aligned} \tag{3.85}$$

Thus, we have the following result:

$$\begin{aligned}\varpi(x,T) &\triangleq \frac{\mathbb{E}_T\left\{\delta_x(X_T)\frac{d\varepsilon_u^{T_-}}{du}\Big|_{u=T_-}\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \\ &= -\rho\left(\Lambda(x,T)+\Phi(x,T)\right)\end{aligned} \tag{3.86}$$

with

$$\Lambda(x,T) = \frac{\sigma_X^2(x,T)\left[\frac{\partial\sigma_X(x,T)}{\partial X}+\frac{\partial^2\sigma_X(x,T)}{\partial X^2}\right]}{2} + \frac{\partial\sigma_X(x,T)}{\partial t} + r_{0,T}\frac{\partial\sigma_X(x,T)}{\partial x} \tag{3.87}$$

and

$$\Phi(x,T) = \frac{\partial\sigma_X(x,T)}{\partial x}\frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T\gamma_{s,T}dB_s\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.88}$$

$$\begin{aligned}\zeta(x,T) &\triangleq \mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T\gamma_{s,T}dB_s\right]\right\} \\ &= \int_0^T\gamma_{s,T}\mathbb{E}_T\left\{\frac{\partial\delta_x(X_T)}{\partial dB_s}\right\}ds \\ &= \int_0^T\gamma_{s,T}\mathbb{E}_T\left\{\delta_x'(X_T)\frac{\partial X_T}{\partial dB_s}\right\} \\ &\sim \Gamma_T\mathbb{E}_T\left\{\delta_x'(X_T)\frac{\partial X_T}{\partial dB_T}\right\}ds \\ &\sim \Gamma_T\rho\mathbb{E}_T\left\{\delta_x'(X_T)\sigma_X(X_T,T)\right\}\end{aligned} \tag{3.89}$$

As:

$$\begin{aligned}\mathbb{E}_T\left\{\delta_x'(X_T)\sigma_X(X_T,T)\right\} &= \int_{y\in\mathbb{R}}\delta_x'(y)\sigma_X(y,T)p_X(y,T)dy \\ &= -\int_{y\in\mathbb{R}}\delta_x(y)\frac{\partial}{\partial y}\left(\sigma_X(y,T)p_X(y,T)\right)dy \\ &= -\left(\frac{\partial p_X(x,T)}{\partial x}\sigma_X(x,t)+\frac{\partial\sigma_X(x,T)}{\partial x}p_X(x,t)\right)\end{aligned} \tag{3.90}$$

with $p_X(x,t) = \mathbb{E}_T\left\{\delta_x(X_T)\right\}$ And finally:

$$\Phi(x,T) \sim -\rho\Gamma_T\frac{\partial\sigma_X(x,T)}{\partial x}\left(\frac{\partial\sigma_X(x,T)}{\partial x}+\sigma_X(x,T)\frac{\partial\ln(p_X(x,T))}{\partial x}\right) \tag{3.91}$$

Summary

$$\sigma^2(x,T) = \sigma^2_{det}(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(x,T)\Gamma_T - \Gamma_T^2 \tag{3.92}$$

$$\sigma^2_{det}(T,K) = \frac{\partial_T C(T,K) - Kf(0,T)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{3.93}$$

$$\tilde{\sigma}^2(T,x) = \frac{2}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)}\int_0^T \gamma_{t,T}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\int_t^T d\epsilon_u\right)dt \tag{3.94}$$

$$\tilde{\sigma}^2(T,x) \approx 2\rho^2\gamma\tau^2\left(1-\left(1+\frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)\left[\Lambda(x,T)+\Phi(x,T)\right] \tag{3.95}$$

Where :

$$\Lambda(x,T) = \frac{\sigma^2(x,T)\left[\frac{\partial\sigma(x,T)}{\partial x} + \frac{\partial^2\sigma(x,T)}{\partial x^2}\right]}{2} + \frac{\partial\sigma(x,T)}{\partial t} + f(0,T)\frac{\partial\sigma(x,T)}{\partial x} \tag{3.96}$$

and

$$\Phi(x,T) = \frac{\partial\sigma(x,T)}{\partial x}\frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T}dB_s^{\mathbb{Q}}\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{3.97}$$

$$\hat{\tilde{\sigma}}^2(x,T) = \tilde{\sigma}^2(T,x) + 2\rho\sigma(x,T)\Gamma_T \tag{3.98}$$

$$\zeta(x,T) \triangleq \mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T}dB_s^{\mathbb{Q}}\right]\right\} \tag{3.99}$$

$$= \int_0^T \gamma_{s,T}\mathbb{E}_T\left\{\frac{\partial\delta_x(X_T)}{\partial dB_s^{\mathbb{Q}}}\right\}$$

$$= \int_0^T \gamma_{s,T}\mathbb{E}_T\left\{\delta_x'(X_T)\frac{\partial X_T}{\partial dB_s^{\mathbb{Q}}}\right\}$$

$$\sim \Gamma_T\mathbb{E}_T\left\{\delta_x'(X_T)\frac{\partial X_T}{\partial dB_T^{\mathbb{Q}}}\right\}$$

$$\sim \Gamma_T \rho \mathbb{E}_T \left\{ \delta'_x(X_T)\sigma(X_T, T) \right\}$$

As:

$$\mathbb{E}_T \left\{ \delta'_x(X_T)\sigma(X_T, T) \right\} = \int_{y\in\mathbb{R}} \delta'_x(y)\sigma(y, T)p_X(y, T)dy$$

$$= -\int_{y\in\mathbb{R}} \delta_x(y)\frac{\partial}{\partial y}\left(\sigma(y, T)p_X(y, T)\right)dy$$

$$= -\left(\frac{\partial p_X(x, T)}{\partial x}\sigma(x, t) + \frac{\partial \sigma(x, T)}{\partial x}p_X(x, t)\right)$$

with $p_X(x, t) = \mathbb{E}_T \{\delta_x(X_T)\}$ And finally:

$$\Phi(x, T) \sim -\rho\Gamma_T \frac{\partial \sigma(x, T)}{\partial x}\left(\frac{\partial \sigma(x, T)}{\partial x} + \sigma(x, T)\frac{\partial \ln(p_X(x, T))}{\partial x}\right) \tag{3.100}$$

3.4.4 Fixed point algorithm

According to the summary above, the algorithm is:

$$\sigma^2_{n+1} = F(\sigma_n), n \in \mathbb{N} \tag{3.101}$$

Where F is the following differential operator:

$$F(\sigma) = \sigma^2_{det}(T, K) - \tilde{\sigma}^2(T, K) - 2\int_0^T \gamma_{s,T}\Gamma_{t,T}dt - 2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\left(\sigma(T, X_T)\right)\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)}ds \tag{3.102}$$

each of the terms:

- $\tilde{\sigma}^2(T, K)$
- $2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}(\sigma(T,X_T))\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)}ds$

depend on the local volatility.

Here is the algorithm in practice:

Input: An initial local volatility surface, a number of maximum number of iterations N,the current number of iterations n, an error ϵ
Output: Hybrid local volatility surface

1. **Initialisation** :
   - Set σ_0 to be the initial local volatility considered
   - n=0
2. **Iteration n**
   While n<N and $norm(\sigma_{n+1} - \sigma_n) > \epsilon$:
   - $\sigma_n = \sigma_{n+1}$,
   - $\sigma_{n+1} = F(\sigma_n)$
3. **Refresh** : n=n+1

Once again, we catch only the adjustment between the hybrid local volatility surface and the deterministic Dupire's local volatility surface that is due to the rates's stochasticity and the skew parameters of the hybrid local volatility surface.

As we have no idea of the structure of the differential operator F (Lipschitz,contraction,...), the aim is to

- Start with a not bad hybrid local volatility
- Correct it in 2 or 3 iterations maximum

Chapter 4

Numerical deployment

4.1 Validation of the Monte Carlo Pricer

As the global method consists of using a Monte Carlo estimator, the first step is to validate the numerical implementation of the Euler scheme of the Model, in terms of:

- Number of time step points
- Number of simulations
- Stability of the scheme

As we saw before, european calls and puts are priced with Black and Scholes closed formulas is the case of flat local volatility surface and Gaussian interest rate.
This is the perfect way to test the Monte Carlo call price with respect to the closed formula price in this context for different model scenarios.

The parameters that we vary in each numerical scenario in the following tests are :

- **The hybrid local volatility function**
  - Case 1: Time dependent function (flat hybrid local volatility)
  - Case 2: A constant surface
- **The model parameters**
  - Case 1: Low volatility rate ($\approx 0.001\%$)
  - Case 2: Medium volatility rate ($\approx 10\%$)
  - Case 3: High volatility rate ($\approx 30\%$)

The aims of these first numerical tests are to validate the Monte Carlo Euler discretisation scheme as well as visualizing numerically the effect of stochastic interest rates on the hybrid model **implied volatility surface**.

4.1.1 Non flat hybrid local volatility

We fixed a time dependent hybrid local volatility and changed the model parameters in order to see the effect of the rates' stochasticity in the model implied smile.

In all the following graphs, we have:

1. *On the top left*:
   - The Monte Carlo call price
   - The closed formula call price
   - The inf bound of the Monte Carlo's confidence interval
   - The sup bound of the Monte Carlo's confidence interval

   In the graphs with log-asset discretisation we add the arbitrage call bounds in order to be sure that the model is arbitrage free.

2. *On the down left*:
   - The error between the Monte Carlo call price and closed formula call price
   - The error between the inf bound and closed formula call price
   - The error between sup bound and closed formula call price

3. *On the top right*: The model theoretical implied volatility smile given by Monte Carlo call prices

4. *On the down right*: The model theoretical implied volatility smile given by closed formula call prices

Test 1: high volatility rate

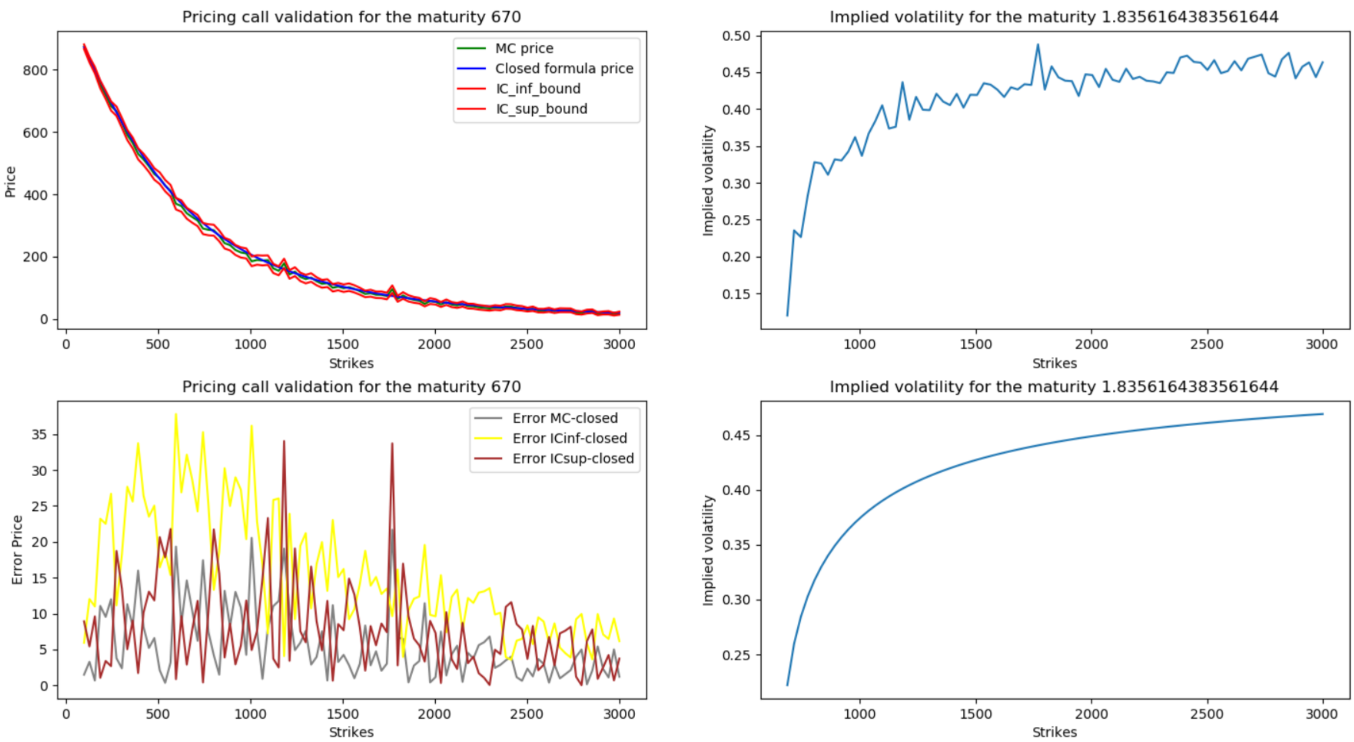

Figure 4.1: Asset discretisation with high volatility rate

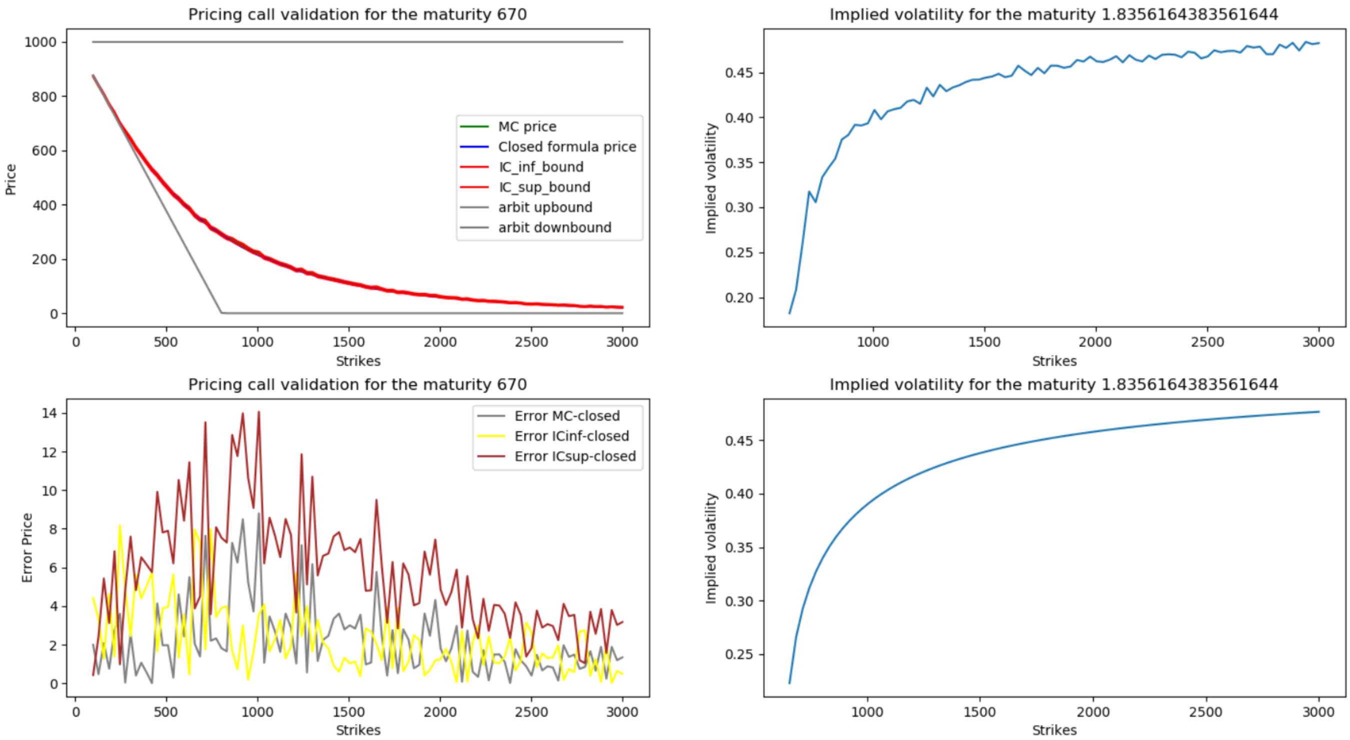

Figure 4.2: log-asset discretisation with high volatility rate

Thus, we see that:

- Both the Monte Carlo call price and the closed formula call price are arbitrage free.
- Visually we see that the confidence interval's width is acceptable even without performing the Richardson Romberg extrapolation. Of course 2RR enhances the estimation quality by a lower width.
- The smile is reproduced by the Monte Carlo price. Here we can notice that the smile is noisy with the effect of the volatility rates So we see clearly the impact of the rates' stochasticity on the model.
- Because of the noise, the smile calibration in an hybrid local volatility model is not perfect and the imperfection increases with the stochasticity effect of rates, in other words, with the volatility parameter of the rates model.

Test 2: low volatility rate

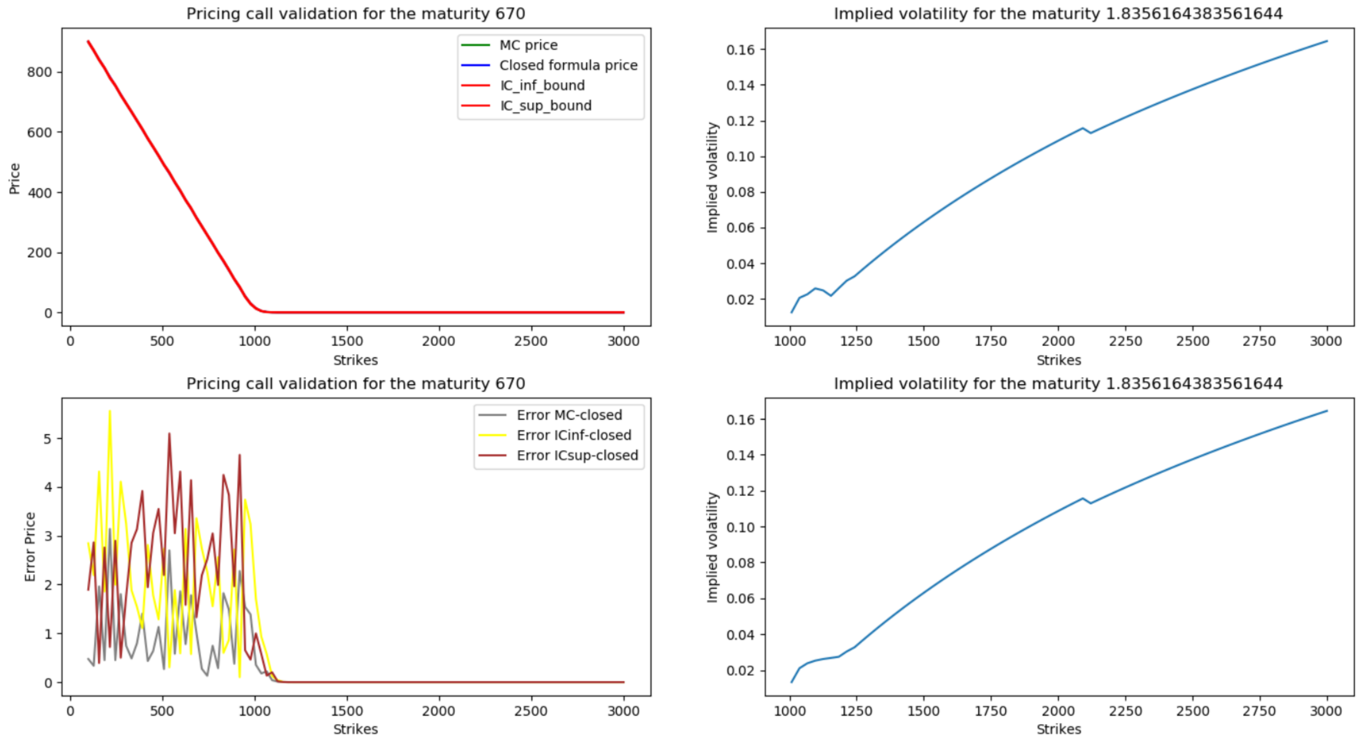

Figure 4.3: Asset discretisation with low volatility rate

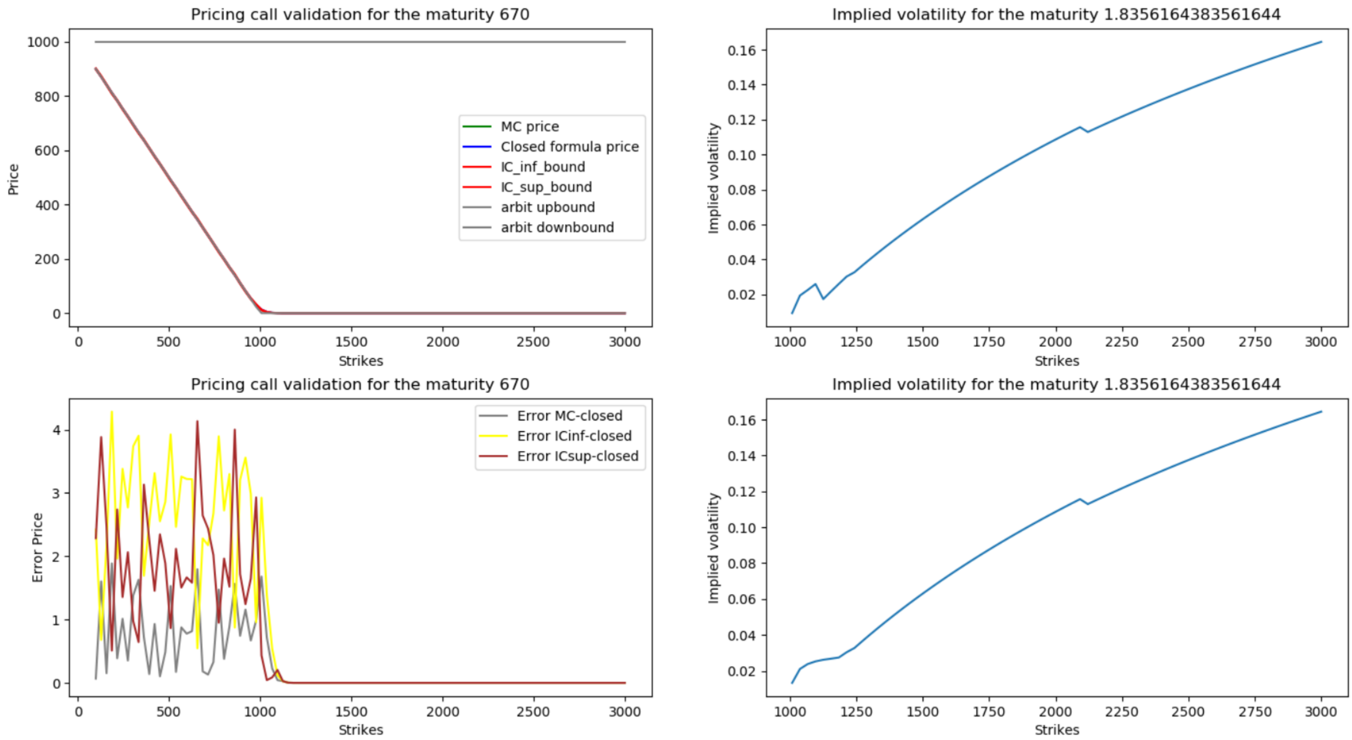

Figure 4.4: log-asset discretisation with low volatility rate

Thus, we see that:

- Both the Monte Carlo call price and the closed formula call price are still arbitrage free.
- Visually we see that the confidence interval's width is acceptable.
- The smile is perfectly reproduced by the Monte Carlo price. Here we can notice that the smile is not anymore noisy because the volatility rate was vanished and we are exactly in one of the cases of 2.1 : r constant (or deterministic time dependent) and σ deterministic time dependent.

But personally speaking, the effect of the additive noise that is done through the volatility rates is much more remarkable in the case of a constant equity volatility surface. That is what will be tackled deeply in the next section.

4.1.2 Constant hybrid local volatility surface

Now, the idea is to be sure that the Black & Scholes model is really encompassed in the hybrid one. Which means to consider a constant local volatility surface and a low volatility rate and to see if the Smile returned will be flat or not. Indeed, it is.

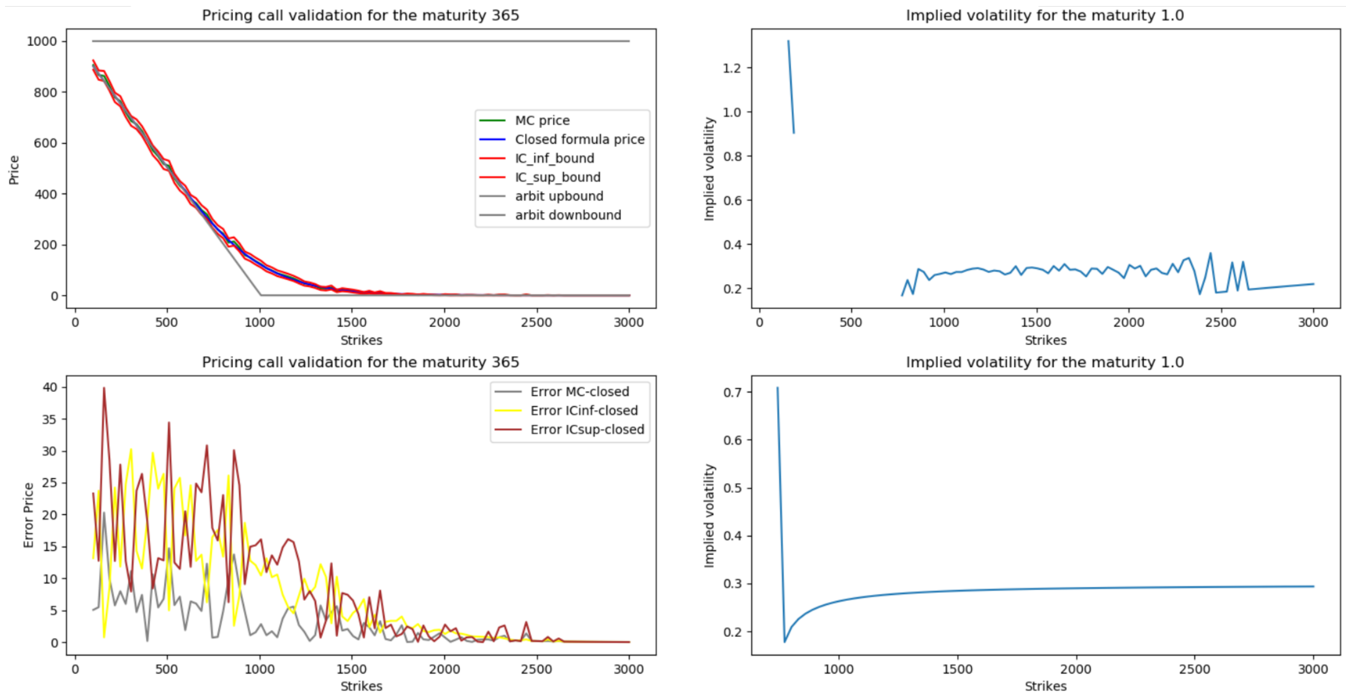

Figure 4.5: Low volatility rate

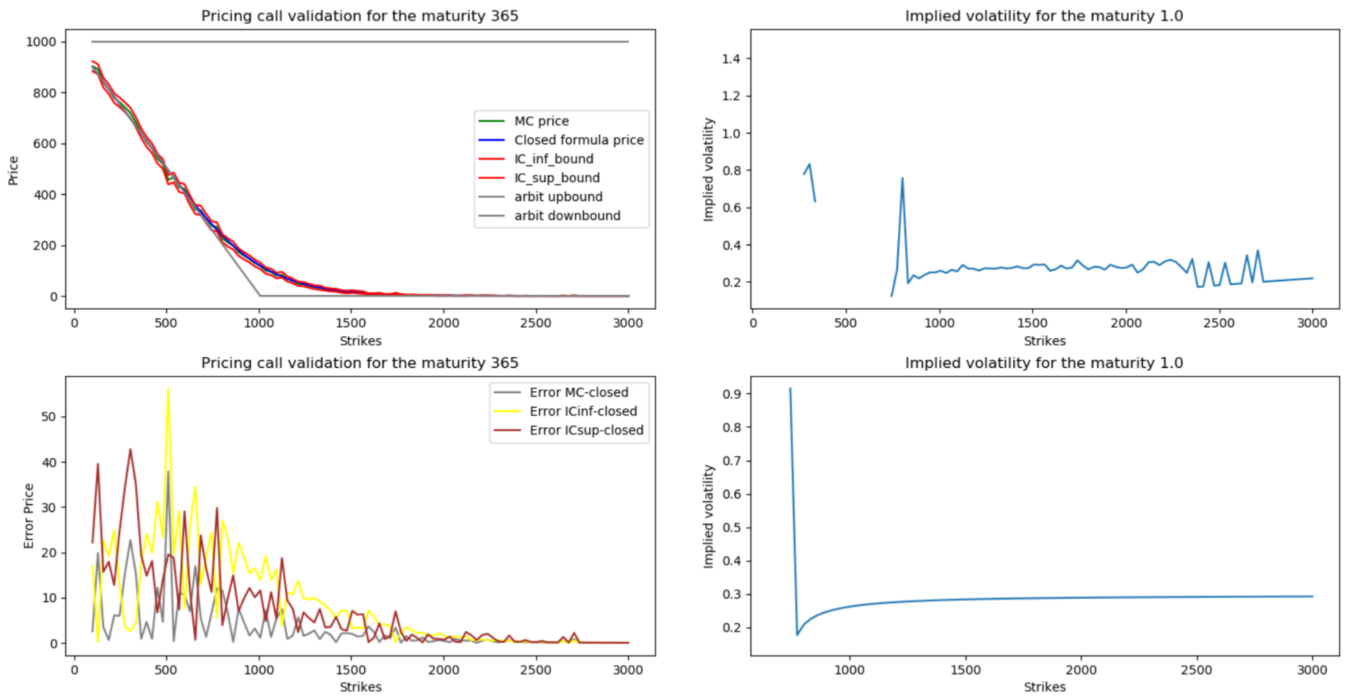

Figure 4.6: Medium volatility rate

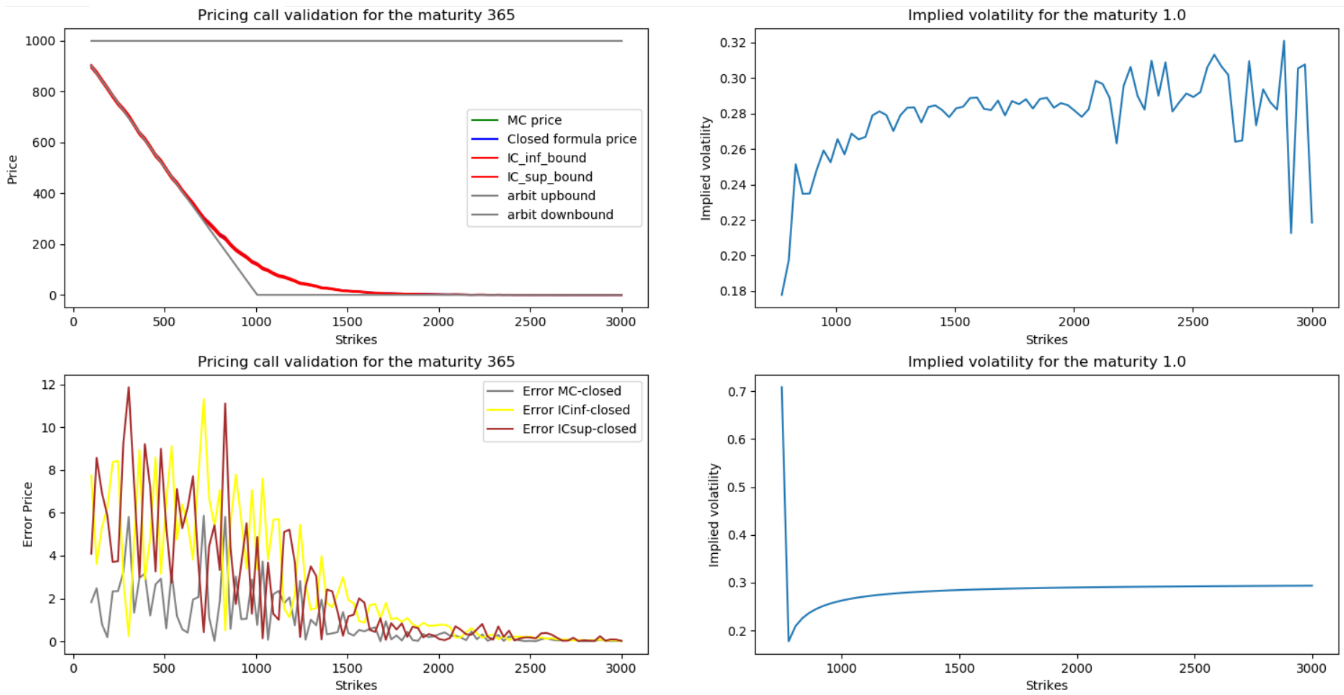

Figure 4.7: High volatility rate

Thus, we see that:

- Both the Monte Carlo call price and the closed formula call price are still arbitrage free.
- Visually we see that the confidence interval's width is acceptable, this time we considered just 100 Monte Carlo simulation numbers and a medium volatility rate level.
- We obtained what was expected, the volatility smile is flat with a higher noise in medium volatility rate level.
- For small strikes, the implied volatility in both cases explodes because in the hybrid local volatility formula the denominator is proportional to K^2 but for reasonable strikes, the implied volatility remains equal in that case to the constant hybrid local volatility.
- For medium volatility rates, we see that the rates' stochasticity is nothing but a noise, which confirms our initial guess and this noise appears in the smile which is greater in the second graph (figure 4.6) than in the first (figure 4.5).
- For high volatility rates, we see the impact of the noise on the equity smile. The noise is considerable as we go far from the money. Around the money, we obtain the expected implied volatility value but the noise is still there with a lower amplitude.
- Here the smile changes because of the adjustment parameters and the deterministic local vol is related to the implied vol by Dupire formula so we see the adjustment effect from theses graphs.

All in all, we conclude numerically that:

- **The volatility rate is in fact a key parameter in the flat local volatility adjustment.**
- **But also the equity-rate correlation factor has its contribution**

4.2 The Monte Carlo local volatility calibration approach in the case of non flat hybrid local volatility surface

As stated before, the hybrid term is approximated by:

$$\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) \approx \frac{1}{M}\sum_{i=1}^{M}\left(r_T^{(i)} e^{-\delta \sum_{k=1}^{N} r_{t_k}^{(i)}} 1_{S_T^{(i)}>K}\right) \tag{4.1}$$

Using the Euler scheme:

$$\begin{cases} S_{t_{k+1}}^{s_{ini}} = S_{t_k}^{s_{ini}}\left(1+r_{t_k}\delta\right) + S_{t_k}^{s_{ini}}\sigma(t_k, S_{t_k}^{s_{ini}})\sqrt{\delta}G \\ r_{t_{k+1}} = r_{t_k} + b(t_k, r_{t_k})\delta + \tilde{\sigma}(t_k, r_{t_k})\sqrt{\delta}\left(\rho G + \sqrt{1-\rho^2}\tilde{G}\right) \end{cases} \tag{4.2}$$

In fact, as the hybrid local volatility is the parameter that we want to calibrate, we have to think about the diffusion parameter that we have to put in the scheme.
As the calibration of the hybrid local volatility is meant to reflect the market, it's wiser to put in the scheme the implied volatility.

4.2.1 Calibration algorithm

Input: Model parameters, Equity smile, Monte Carlo parameters
Output: Hybrid local volatility surface

1. Perform interpolation of the Smile by strike and maturity
2. Estimate all the partial derivatives by finite differences
3. Perform the Monte Carlo computation of the hybrid term
4. Agregate all the terms
5. Perform grid interpolation
6. Return the hybrid local volatility surface

Smile interpolation

We consider as an input an equity smile with discrete strikes and maturities. Thus, in order to perform the computation of the strike and maturity partial derivatives we have to interpolate in strike and maturity the equity smile so as to obtain consistant partial derivatives.
Several interpolation approaches were tested.

In this section, we denote $\mathbb{T}$ and $\mathbb{K}$ the set of all maturities and strikes available in the equity smile.

- Linear interpolation
The linear interpolation was performed for each strike to interpolate in maturity. That is to say :

$$\forall T \in \mathbb{T}, \hat{\sigma}(T,.) = \sum_{K\in\mathbb{K}} L(T,.)\sigma(T,.) \tag{4.3}$$

Where L in the Lagrange interpolation polynomial function.
The ideas that were tested were to interpolate per strike and maturity the total variance $T \longrightarrow T\sigma^2(T,.)$

and $K \longrightarrow K\sigma^2(.,K)$ instead of the implied volatility as we expect that the variance is smooth enough.

- L^2 interpolation
From a smoothness point of view, we experienced Hermite polynomial interpolation rather than just a simple linear interpolation as the functional interpolated won't be C^2 in the points of the interpolation. We recall the definition of Hermite polynomial functions:

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{\frac{-x^2}{2}}}{dx^n} \tag{4.4}$$

Thus, the interpolated volatility function is:

$$\forall T \in \mathbb{T}, \hat{\sigma}(T,.) = \sum_{n=1}^{N} \alpha_n H_n(.) \tag{4.5}$$

Where $(\alpha_n)_{n\in\{1,\dots N\}}$ are determined by least square minimisation.

- Stineman interpolation
It is a monotone convex interpolation scheme.
Given a set of points $(x_i, y_i)_{i\in I}$, we interpolate iteratively between each (x_i, y_i) and (x_{i+1}, y_{i+1}) following the local monotocicity and convexity of the function.
We denote:

- The line segment with equation:

$$y_0 = y_j + s_j\,(x - x_j) \tag{4.6}$$

- The local slope of the curve:

$$s_j = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} \tag{4.7}$$

- The vertical distance Δy_j:

$$\Delta y_j = y_j + y_j^{'}\,(x - x_j) - y_0 \tag{4.8}$$

- The vertical distance Δy_{j+1}:

$$\Delta y_{j+1} = y_{j+1} + y_{j+1}^{'}\,(x - x_{j+1}) - y_0 \tag{4.9}$$

The interpolation is performed using the following method:

- If $\Delta y_j \Delta y_{j+1} > 0$:

$$y = y_0 + \frac{\Delta y_j \Delta y_{j+1}}{\Delta y_j + \Delta y_{j+1}} \tag{4.10}$$

- If $\Delta y_j \Delta y_{j+1} < 0$: There must be an inflection point between x_j and x_{j+1} and :

$$y = y_0 + \frac{\Delta y_j \Delta y_{j+1}\,(2x - x_j - x_{j+1})}{(\Delta y_j - \Delta y_{j+1})\,(x_{j+1} - x_j)} \tag{4.11}$$

All the previous interpolation methods are given as a choice in the code.

Partial derivative computation

The computation of the partial derivatives is made by following theses steps:

1. Perform an interpolation per strikes and maturities of the equity smile

2. Starting from the equity smile we compute the related call Market prices by:

$$C_{mkt}(T,K) = C_{BS}(T,K,\sigma_{impl}(T,K)) \tag{4.12}$$

3. In order to compute the partial maturity derivative we use the interpolation per strikes and set the following centred scheme:

$$\partial_T C(T,K) \approx \frac{C_{BS}(T+\delta T,K,\hat{\sigma}_{impl}(T+\delta T,K)) - C_{BS}(T-\delta T,K,\hat{\sigma}_{impl}(T-\delta T,K))}{2\delta T} \tag{4.13}$$

And by interpolating for each maturity:

$$\partial_{KK} C(T,K) \approx \frac{C_{BS}(T,K+\delta K,\hat{\sigma}_{impl}(T,K+\delta K)) + C_{BS}(T,K-\delta K,\hat{\sigma}_{impl}(T,K-\delta K))}{\delta K^2} \frac{-2C_{BS}(T,K,\hat{\sigma}_{impl}(T,K))}{\delta K^2} \tag{4.14}$$

Grid interpolation

After computing all the terms of (3.25), we perform a grid interpolation as the output of (3.25) computation is for the same discrete strikes and maturities of the input equity smile. And we get the hybrid local volatility surface.

4.2.2 Numerical results

We consider the following context:

- Gaussian rates (Hull & White model)
- The same volatility rate scenarios

High volatility rates

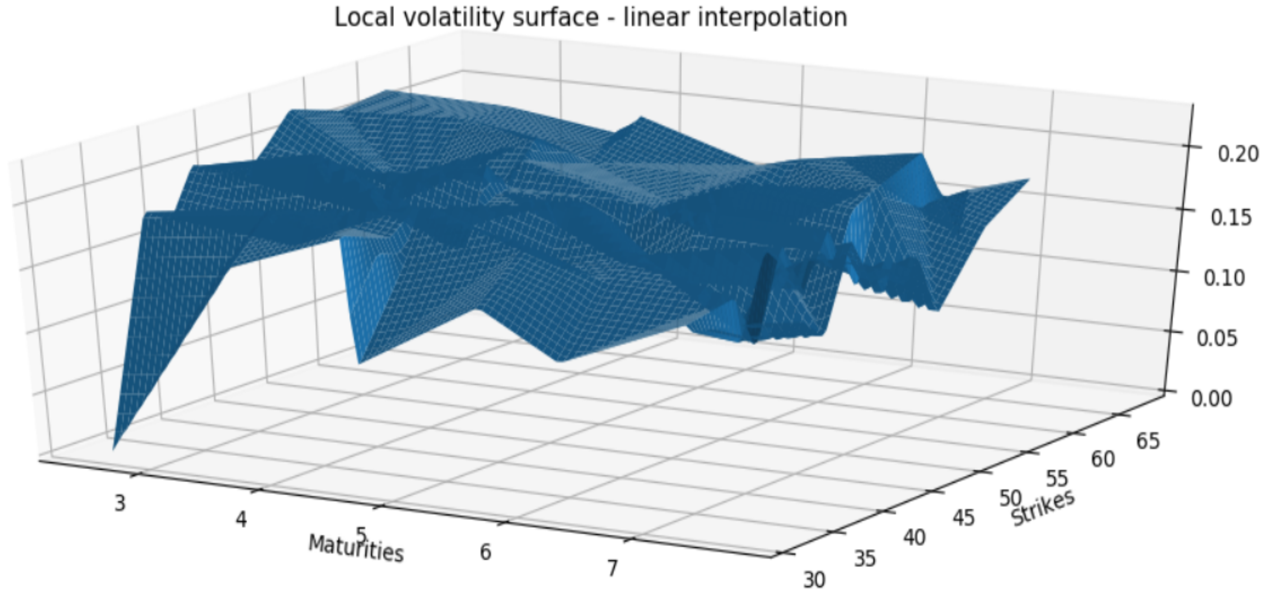

Figure 4.8: Hybrid local volatility surface with non flat equity smile and high rate vol

Once again, we notice the noise produced by the rates's stochasticity.

Indeed, a single hybrid local volatility surface is meaningless without being sure to reproduce classical local volatility features. For that reason, we consider the case of low volatility rates level in order to collapse the adjutment term's stochasticity and see if we can reproduce a flat equity smile.

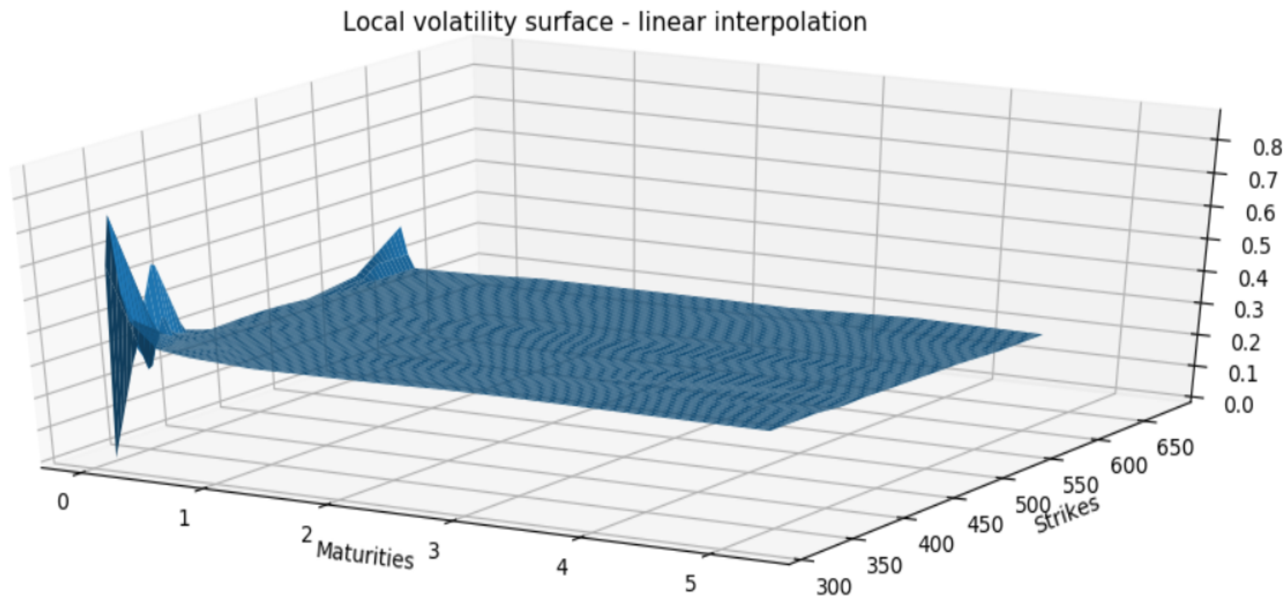

Figure 4.9: Hybrid local volatility surface with flat equity smile and low rate vol

Thus, the hybrid local volatility is as expected. For small maturities and strikes some boundary effects from the grid interpolation method (here linear) can appear but we obtain for reasonable maturities and strikes that the hybrid local volatility is exactly the equity flat smile (valued 0.3 for all strikes and maturities).

This is also reproduced by the Dupire's coding engine (coded separatly as a local volatility benchmark):

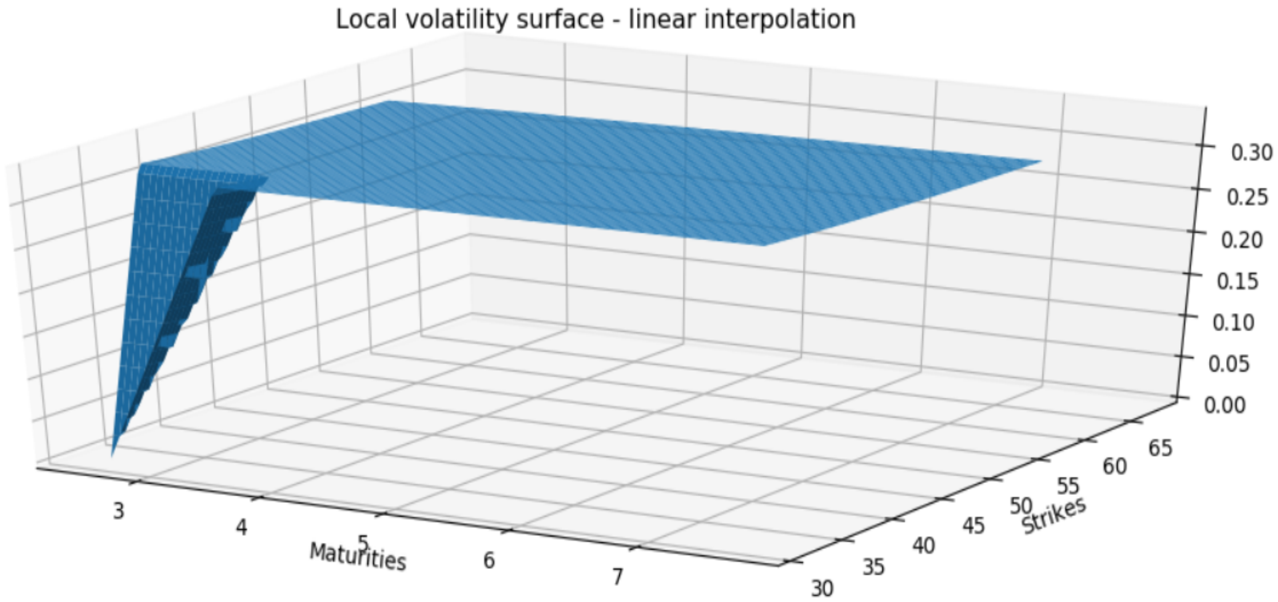

Figure 4.10: Dupire's local volatility surface with flat equity smile

Thus, the hybrid local volatility surface represents really **what is expected**:

- A sensitivity to the volatility rate
- A sensitivity to the equity-rate correlation

But still the quantification of the rate's stochasticity is not controlled. Which is the main strength of the hybrid local volatility expansion.

Before tackeling the "PDE expansion" part, we must test it with a given arbitrage-free local hybrid local volatility surface, which will be the object of the next section

4.3 Local volatility surface construction

A good local volatility surface is a volatility surface whose properties are close to an arbitrage-free implied volatility surface.
Globally, we expect from a good local volatility surface to be:

1. **Polynomial in log-moneyness in wings, ie:**

$$\limsup_{x\to\infty} \frac{\sigma(T,x)^2 T^{\tilde{\beta}}}{x} < \infty \tag{4.15}$$

2. **To have a mean reverting ATM term:**

$$\sigma(T,x) \underset{+\infty}{\sim} \frac{c}{T^{\beta}} \tag{4.16}$$

Where c and β are respectively constants in $\mathbb{R}$.

Actually the most important property to conserve is the first one.

In order to have a meaningful verification of the "PDE expansion", we opted for random local volatility generators.

4.3.1 Othmane ZARHALI's random local volatility generator

The approach I adopted is the same approach explained in (4.9) and (4.10). The local volatility decomposition is the following:

$$\begin{cases} \sigma(T,x) = \sigma_{ATM}(T,x) + \left(\frac{0.8}{T^{0.24}}((G_1\left(\frac{K}{S_0}\right)^2) + 1) + \frac{0.4}{T^{0.26}}(G_2 e^{-K}\left(\frac{K}{S_0}\right)^2 + 1)\right) \\ \sigma_{ATM}(T,x) = G_3\frac{0.8}{T^{0.27}} + std(2) * \frac{G_4}{T^{0.24}} + \frac{0.7}{T^{0.26}} \\ x = log\left(\frac{KD(0,T)}{S_0}\right) \end{cases} \tag{4.17}$$

Where:

- $(G_i)_{i\in\{1,2,3,4\}}$ are independent uniform random variables.
- std is a standard deviation decreasing function:

$$std(n) = e^{-0.6n^3} \tag{4.18}$$

- $D(0,T)$ is the discount factor between 0 and T.

4.3.2 Arnaud RIVOIRA's random local volatility generator

The approach here is a little bit different, the aim is to have a local volatility surface quasi-linear in the wings and as smooth as possible. It's a kind of normalised exponential random function:

$$\sigma(T,x) = \lambda e^{-\eta H(x,T)} \tag{4.19}$$

Where:

- H is a random bivariate polynomial function.
- λ is a normalisation factor to ensure that the random volatility surface is bounded in $[0,1]$.
- η is a mean reverting coefficient.

Both random local volatility surface generators where tested in the next section.

4.4 "PDE expansion"

We recall the main formulas:

$$\sigma^2(T,x) = \sigma^2_{det}(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(T,x)\Gamma_T - \Gamma_T^2 \tag{4.20}$$

$$\tilde{\sigma}^2(T,x) \approx 2\rho^2\gamma\tau^2\left(1-\left(1+\frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)[\Lambda(x,T)+\Phi(x,T)] \tag{4.21}$$

It appears from the formulation that the hybrid local volatility skew dependence and the rate model dependence are quiet separated.
There is also a number of relevant remarks:

- The formula allows to control the hybrid local volatility calibration by the adjustment of the rate's parameters and the skew terms
- The previous formula is symmetric: given the deterministic local volatility we can determine the hybrid local volatility and vice-verca.

4.4.1 "PDE expansion" - term analysis

- Actually, $\sigma^2_{det}(T,x)$ is non compressible.
- The term Γ_T^2 looks as follows for a volatility rate 1%:

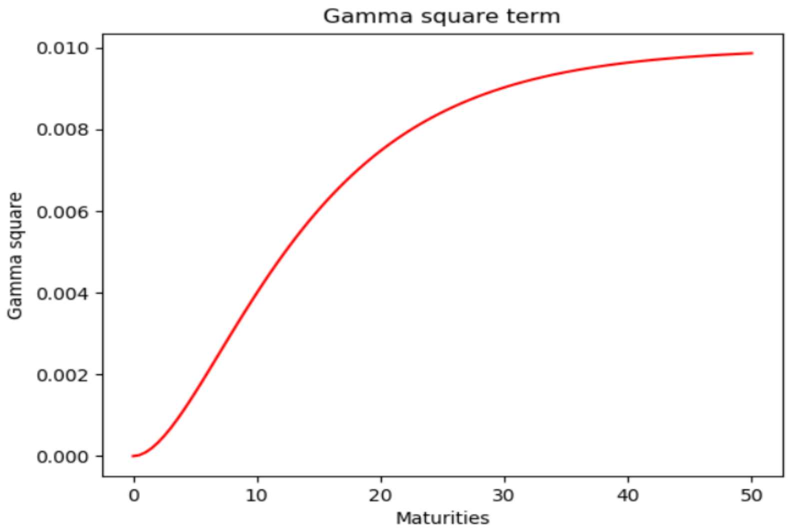

Figure 4.11: Gamma square

Remark 3 *There is an important point to highlight:*
In the steady state, the Γ_T^2 *term is proportional to* $\left(\frac{\gamma}{\alpha}\right)^2$*. Thus, for a really high volatility rate* ($\gamma \sim 30\%$) *and for usual mean reversion* ($\alpha \sim 0.1$)*, the term* Γ_T^2 *could be proportional to* 4 *in the steady state. As a result, the correction term of the formula (4.10) could not be a correction term anymore.*

The adjustment $\tilde{\sigma}^2(T,x)$ term depends on the hybrid local volatility's regularity:

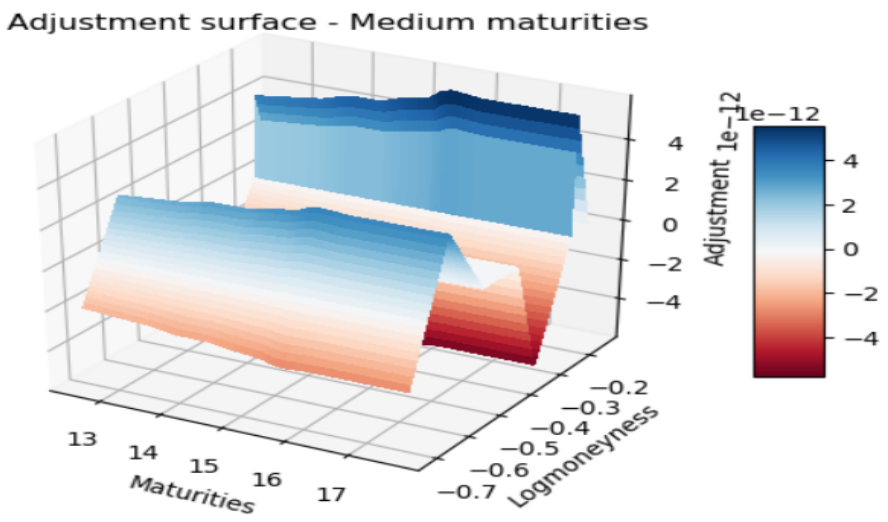

Figure 4.12: Adjustment term

For long maturities (~ 20 years):

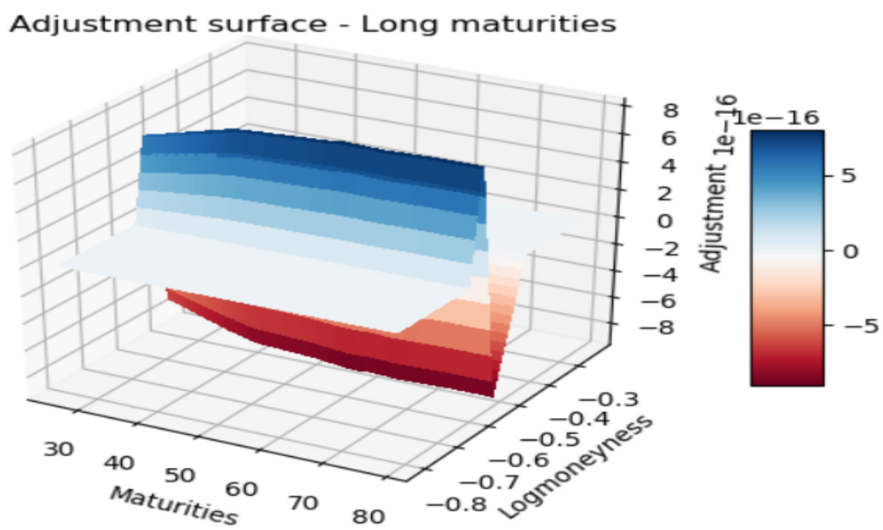

Figure 4.13: Adjustment term

4.4.2 Fixed point algorithm

Let's find out if the classical stylized facts are reproduced by the method.
We consider the following scenarios:

- **Scenario 1**: Vanishing correlation and volatility rate vs flat hybrid local volatility
- **Scenario 2**: Vanishing correlation and volatility rate vs non flat hybrid local volatility
- **Scenario 3**: Non vanishing correlation and volatility rate vs flat hybrid local volatility
- **Scenario 4**: Non flat hybrid local volatility

Here are the respective expectations and the results:

Scenarios	Expectations	Results
1	The same flat hybrid local volatility	✓
2	A corrected flat hybrid local volatility	✓
3	A corrected hybrid local volatility	✓
4	A corrected non flat hybrid local volatility	✓

Let's present some numerical simulations.

Scenario 1:

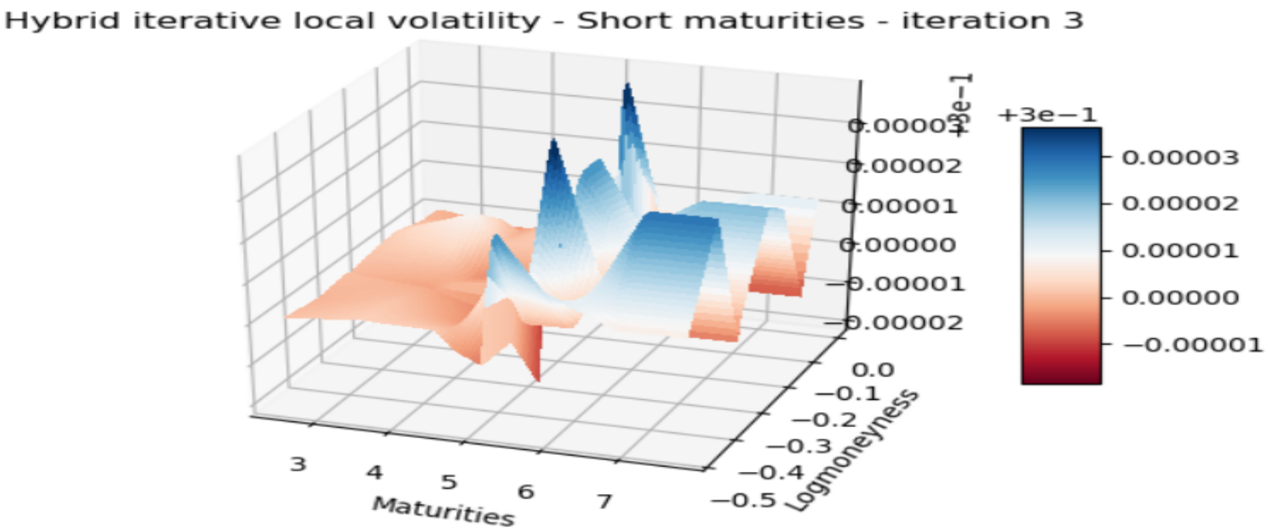

Figure 4.14: Output hybrid local volatility

Scenario 3:

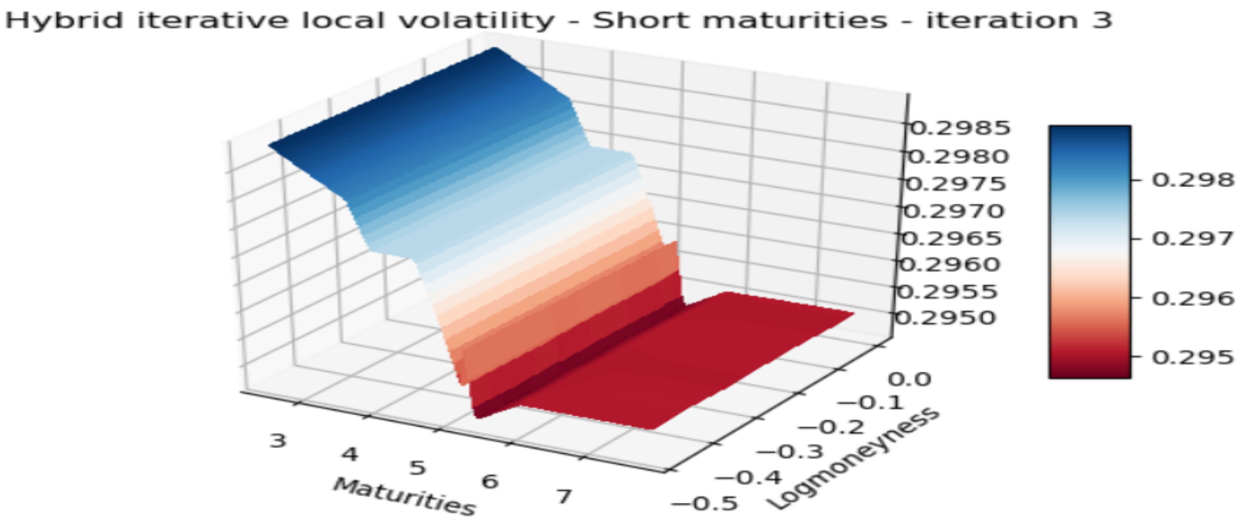

Figure 4.15: Output hybrid local volatility

As a result, we observe that:

- For a completely constant hybrid local volatility surface, the output of the expansion is exactly the same. No correction was operated
- For a non flat initial hybrid local volatility, the correction of the deterministic local volatility was operated thanks to the skewness of the hybrid local volatility over the iterations and to the rate's model parameters. We see that in the second figure, the output is a correction of the deterministic (Dupire's) local volatility.

Chapter 5

Conclusion

We can sum up all the relevant results of the intern in the following key points:

- The aim of the intern was to quantify the impact of the stochastic interest rates on the local volatility
- The toy flat local volatility model was important to construct the initial guesses
- Some analogies have been found out between the flat and non flat local volatility (ρ, and γ adjustment dependence,...)
- The Monte Carlo direct computation does not provide much insight on the effective impact of the rate's parameters. In terms of time computation, the latter is high as much as the equity dimension is high (multiasset diffusions).
- The "PDE expansion" allows to correct iteratively the hybrid local volatility, but some regularity assumptions on the hybrid local volatility and some empirical conditions on the stochastic interest rate's parameters are required.

All in all, this intern was an opportunity for me to sharpen my quantitative skills in stochastic analysis and local volatility models as well as the underlying computing skills related to the IT aspects of the intern.

Appendix A

Malliavin calculus

In this part, we will present the main results concerning the Malliavin calculus and the integration by parts formula.

A.1 Malliavin derivative

A.1.1 Malliavin derivative for piecewise constant Wiener processes

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ a filtred probability space where we define a classical Browian motion denoted (W_t) and the $\mathscr{F}_t$ is its canonical one.

We will define the Malliavin derivative for piecewise constant wiener processes and then extend its definition to functionnals of continuous Wiener processes and to diffusions by density arguments.

We denote the dyadic subdivision $t_k^n = \frac{k}{2^n}$ for $k = 1, .., n$.
And also the brownian increment $\Delta_k^n = W_{t_{k+1}^n} - W_{t_k^n}$ pour $k = 1, .., n$.

Definition 1 *We consider the following set of functionals :*

$$S_n = \{f(\Delta_0^n, ..., \Delta_{2^n-1}^n), f \in C_p^\infty(\mathbb{R}^{2^n})\} \tag{A.1}$$

The set of peacewise Wiener functionals is then $\cup_{n>0} S_n$

Definition 2 *We consider the set of the following processes :*

$$P_n = \{\sum_{i=0}^{2^n-1} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(t) F_i, F_i \in S_n\} \tag{A.2}$$

We denote the set of simple processes $\cup_{n>0} P_n$

We will now define the Malliavin derivative of simple processes

Definition 3 *Given a functional $F \in S$, we denote $F = f(\Delta^n)$.*
The Malliavin derivative *of F is defined by:*

$$D_s F = \sum_{i=0}^{2^n-1} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(s) \frac{\partial f}{\partial x_i}(\Delta^n) \tag{A.3}$$

Hence, the Malliavin derivative of W_1 is : $D_s W_1 = \mathbb{1}_{[0,1[}(s)$.

We will then define the adjoint operator of Malliavin derivative called the Skorohod operator.

Definition 4 *Given the following simple process u:*

$$u = \sum_{i=0}^{2^n-1} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(s) f_i(\Delta^n) \tag{A.4}$$

The Skorohod operator $\delta : P_n \longrightarrow S_n$ *is defined as :*

$$\delta(u) = \sum_{i=0}^{2^n-1} f_i(\Delta^n)\Delta_i^n - \sum_{i=0}^{2^n-1} \frac{\partial f_i}{\partial x_i}(\Delta^n)\frac{1}{2^n} \tag{A.5}$$

Remark 4 *We notice that if u is adapted, ie f_i is $\mathscr{F}_{i-1}$ measurable, the second term with partial derivatives will vanish and the Skorohod integral coincides with Itô integral*

A.1.2 Extension of Malliavin derivatives to square integrable processes

We will extend the notion of Malliavin derivative and Skorohod integral to stochastic processes of interest for us which are L^2 ones.
We have the following bridge result :

- S is dense in L^2
- P is dense in $L^2([0,1])$

Thus, the key missing element in order to perform the extension is a closing property on Malliavin and Skorohod operators.
Indeed :

Property 1 *The operators D et δ are closed ones, ie: for a given sequence of simples processes F_n that tends to 0 in L^2 sense, if $(DF_n)_n$ tends to u in $L^2([0,1])$ then $u = 0$.*

The extension is performed in the following way :

Definition 5 *Let $\mathbb{D}^{1,2}$ the set of the following processes :*

$$\mathbb{D}^{1,2} = \{F \in L^2, \exists (F_n) \in S : F_n \xrightarrow[n\infty]{L^2} F, DF_n \xrightarrow[n\infty]{L^2} u\} \tag{A.6}$$

We assume then that $DF = u$, which defines the Malliavin derivative for square integrable processes. The Skorohod integral is extended with the same idea.

A.2 Malliavin derivative in the Wiener space

We denote respectively (.,.) and $\|.\|$ the scalar product and its norm in L^2 and $\langle .,. \rangle$ the canonical scalar product in $\mathbb{R}^d$. Let's tackle the multidimensional case.

Definition 6 *We define functionals of Wiener processes :*

$$F = f(\delta(h_1), ..., \delta(h_n)) \tag{A.7}$$

Where :

- *f in C^{∞} valued in $\mathbb{R}^n$*
- *$(h_1, ..., h_n)$ in L^2 valued in $\mathbb{R}^n$*
- *$\delta(h_i)$ Wiener integral associated to h_i :*

$$\delta(h_i) = \int \langle h_i(t), dW_t \rangle \tag{A.8}$$

For the functionaals (A.7), we define there Malliavin derivative.

Definition 7 *The Malliavin derivative of F is :*

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\delta(h_1), ..., \delta(h_n)) h_i(t) \tag{A.9}$$

The Skorohod operator is the adjoint of Malliavin operator in a way that for any square integrable process u :

$$\mathbb{E}\left(F\delta(u)\right) = \mathbb{E}\left(\int \langle u_t, D_t F \rangle \, dt\right) \tag{A.10}$$

This is a key formula in stochastic analysis. It's analoguous to the classical analytical integration by parts for Riemann integrals.
We notice once again that if $(u_t)_t$ is $\mathscr{F}_t$-adapted then the Skorohod integral is an Itô one.

Remark 5 *According to the simple processes Malliavin derivative, we said that for an $\mathscr{F}_t$-adapted simple process F, we have for all $s>t$, $D_s F = 0$. This is conserved by density . Then, for all square integrable processes $\mathscr{F}_t$-adapted, $D_s F = 0$.*

We preesent some intuitive Malliavin derivation technics and the so called the Malliavin integration by part formula.

Property 2 *Let F et G be two Wiener functionals and h a square integrable function.*
Indeed, we have the following properties in dimension 1 that can be generalised in a multidimensional case :

- $\delta(hF) = F\delta(h) - \int h(s) D_s F ds$
- $D_t(FG) = F D_t(G) + G D_t(F)$

We have also these commutation formulas that are useful in practice :

- $D_t\left(\int_0^T u_s dW_s\right) = u_t + \int_t^T D_t\left(u_s\right) dW_s$ *for all $T>0$*
- $D_t\left(\int_0^T u_s ds\right) = \int_t^T D_t\left(u_s\right) ds$ *for all $T>0$*
- *Compound derivatives:*
  $D_s\left(\phi(F)\right) = \phi'(F) D_s\left(F\right)$

It's easy to verify them in the case of simple processes.

A.3 Malliavin derivative for diffusions

Definition 8 *Let X^x an unidimensional diffusion starting from x, as a string solution of the following SDE :*

$$dX_t^x = b(t, X_t^x)dt + \sigma(t, X_t^x)dW_t \tag{A.11}$$

Where b and σ are $C^{1,2}$.
*We define the **tangent process** associated to X^x denoted Y by :*

$$Y_t = \frac{\partial X_t^x}{\partial x} \tag{A.12}$$

Remark 6 *We notice that the tangent process is a strong solution of the SDE :*

$$dY_t = Y_t \left(\frac{\partial b}{\partial x}(t, X_t^x)dt + \frac{\partial \sigma}{\partial x}(t, X_t^x)dW_t \right) \tag{A.13}$$

Here is the interesting result that we will prove using the previous arguments :

Proposition 1 *For a given diffusion X its Malliavin derivative is :*

$$\forall s > 0, D_s X_t = Y_t Y_s^{-1} \sigma\left(s, X_s^x\right) \mathbb{1}_{s \leq t} \tag{A.14}$$

Proof 1 *Starting from the dynamic of X^x:*

$X_t^x = x + \int_0^t b(u, X_u^x)du + \int_0^t \sigma(u, X_u^x)dW_u$

We apply the Malliavin operator to this expression. By using the commutation properties, we obtain :

$D_s X_t^x = \int_s^t D_s b(u, X_u^x)du + \int_s^t D_s \sigma(u, X_u^x)dW_u + \sigma(s, X_s^x)$

And using the compound derivation leads to :

$D_s X_t^x = \int_s^t D_s X_u^x \frac{\partial b}{\partial x}(u, X_u^x)du + \int_s^t D_s X_u^x \frac{\partial \sigma}{\partial x}(u, X_u^x)dW_u + \sigma(s, X_s^x)$

Thus, $(D_s X_t^x)_t$ satisfies the same tangent process's SDE with different initial points so we can assume that :

$D_s X_t^x = \lambda Y_t \mathbb{1}_{s \leq t}$

and by identification, we get $\lambda = Y_s^{-1}\sigma(s, X_s^x)$

Remark 7 *We can conclude this appendix by some useful remarks:*

- *For anticipative diffusions we still conserve the property :*
  $D_s X_t^x = 0$ *if s greater than t.*
- *(A.14) assumes that if we have a closed formula of the process X^x and its tangent process, then its associated Malliavin derivative has also a closed formula. Otherwise, its Malliavin derivative will satisfy the SDE (A.13).*

Appendix B

Fokker-Planck PDE

The aim of this appendix is to give a short overview of Fokker Planck PDE and its link with diffusion processes.
We consider the following diffusion process:

$$\begin{cases} dX_t^{X_0} = b(t, X_t^{X_0})dt + \sigma(t, X_t^{X_0})dW_t \\ X_0^{X_0} = X_0 \in \mathbb{R}^n \end{cases} \tag{B.1}$$

Where both b and σ are Lipschitz continuous and X_0 a random variable valued in $\mathbb{R}^n$. The Fokker-Planck PDE is a PDE satisfied by the conditional density of X_t^x denoted: $f : (x, t) \longrightarrow P(x, t|x_0, t_0)$. in our case $t_0 = 0$.

Property 3 *Fokker-Planck PDE is:*

$$\begin{cases} \partial_t f(t,x) = -\sum_{i=1}^n \partial_{x_i} \left(b_i(t,x) f(t,x) \right) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i,x_j} \left(a_{i,j}(t,x) f(t,x) \right) \\ f(0,.) = f_0(.) \end{cases} \tag{B.2}$$

Where:

- $a_{i,j}(t,x) = \sum_{k=1}^n \left(\sigma_{i,k} \sigma_{k,j} \right)(t,x)$
- $\frac{d\mathbb{P}_{X_0}(x)}{\lambda(dx)} = f_0(x)$

This PDE can be solved using finite differences for example.

Appendix C

Dupire's local volatility surface

The goal of this appendix is to justify theoretically the numerical simulations of (4.1.1) by demonstrating the link between Dupire's local volatility with deterministic rates and the equity's smile (implied volatility).

C.1 Dupire's formula for deterministic interest rates

We recall the hybrid local volatility models with stochastic rates:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{C.1}$$

Where:

$$\sigma^2_{det}(T,K) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{C.2}$$

In the case of deterministic time dependent rates, r_T goes out of the expectation and we have the Dupire's local volatility for time dependent rates:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) + \frac{K r_T \partial_K C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{C.3}$$

Thus :

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) + K r_T \partial_K C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{C.4}$$

We introduce the implied volatility by its basic definition:

$$C(T,K) = C_{BS}(T,K,\sigma_{imp}(T,K)) \tag{C.5}$$

And we obtain by direct computations the following property.

C.2 Local volatility in terms of implied volatility

Property 4 *The local variance $v : (T,K) \longrightarrow \sigma^2(T,K)$ is expressed in function of the total implied variance $w : (T,K) \longrightarrow T\sigma^2_{imp}(T,K)$ by:*

$$v(T,K) = \left(\frac{\partial_T w}{1 + \frac{K}{w}\partial_K w + \frac{1}{2}\partial_{KK} w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{K^2}{w}\right)(\partial_K w)^2} \right)(T,K) \tag{C.6}$$

Thus, in the hybrid local volatility surface, controlling the hybrid local volatility function and fixing the model parameters allowed to see directly its impact on the equity smile's deformation.

C.3 Particular test case

From the previous formula, we have:

$$v(T,K) = \left(\frac{\sigma_{imp}^2 + 2T\sigma_{imp}\partial_T\sigma_{imp}}{1 + \frac{K}{w}\partial_K w + \frac{1}{2}\partial_{KK} w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{K^2}{w}\right)(\partial_K w)^2} \right)(T,K) \tag{C.7}$$

If we consider a flat equity smile (a constant implied volatility for all strikes and maturities), we notice that the local variance coincides with the implied variance. That must be verified numerically (see Figure page 30 and 31).

Appendix D

Hybrid local volatility expansion: The risk neutral proof

D.1 Objective

The aim of this appendix is to demonstrate the dynamics of the hybrid equity-rate local volatility in the case of an equity and stochastic interest rates model. The dynamic is actually a PDE dynamical system that we will demonstrate. The proof was entirely performed by myself from scratch.

D.2 Local volatility expression in the case of stochastic interest rates

D.2.1 Preliminaries

It's well known that if we take into account the rates's stochasticity, the local volatility is :

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) - K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.1}$$

The proof is a pure application of the Itô-Tanaka formula on the actualised call payoff.

As we consider risk neutral gaussian interest rates :

$$r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}} \tag{D.2}$$

There is an interesting formula between the rates's parameters and the zero coupon bond which will lead to an expression of the hybrid local volatility as a shift of a Dupire's one.

We have by the simple definition of the forward spot rate :

$$f(0,t) = -\partial_t ln\left(B(0,t)\right) \tag{D.3}$$

According to (2), the interest rates have risk neutral Gaussian dynamics, so for all t>0, we have :

$$B(0,t) = e^{-\int_0^t \mu_s ds + \frac{1}{2}\int_0^t \Gamma_{s,t}^2 ds}$$

Where : $\Gamma_{s,t} = \int_s^t \gamma_{s,u} du$.

By direct computations, we obtain :

$$f(0,t) = \mu_t - \frac{1}{2}\left(\int_0^t \gamma_{s,t} ds\right)^2 \tag{D.4}$$

In particular for each fixed maturity T :

$$f(0,T) = \mu_T - \frac{1}{2}\left(\int_0^T \gamma_{s,T} ds\right)^2 \tag{D.5}$$

D.2.2 Hybrid local volatility formula

Thus, the expression (1) becomes:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) - K\mu_T \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} - \frac{K\mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right] e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.6}$$

By using (5), we obtain :

$$\sigma^2(T,K) = \frac{\partial_T C(T,K) - Kf(0,T) \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} - \frac{K\mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right] e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} - \left[\int_0^T \gamma_{s,T} ds\right]^2 \frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.7}$$

We have the following decomposition:

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \frac{\mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right] e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} - \left[\int_0^T \gamma_{s,T} ds\right]^2 \frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.8}$$

Where σ^2_{det} is the Dupire's local volatility with time dependent interest rates the forward spot rate $t \longrightarrow f(0,t)$:

$$\sigma^2_{det}(T,K) = \frac{\partial_T C(T,K) - Kf(0,T) \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.9}$$

P.S : *The value of $f(0,t)$ for each maturity t depend only on the zero coupon bond quotes.*

As a result, σ^2_{det} can be perfectly computed as a standard Dupire's local volatility.
The main difficulty is the hybrid term.

Furthermore, the following term: $\left[\int_0^T \gamma_{s,T} ds\right]^2 \frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)}$ can be calibrated to call prices as such :

$$\left[\int_0^T \gamma_{s,T} ds\right]^2 \frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} \delta_K(S_T^{s_{ini}})\right)} = \frac{[\Gamma_T]^2}{K}\left(\frac{-\partial_K C(T,K)}{\partial_{KK} C(T,K)}\right) \tag{D.10}$$

Where:

- $\Gamma_T = \int_0^T \gamma_{u,T} du$

D.2.3 Hybrid local volatility decomposition

The stochastic integral $\int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}}$ is actually a skorohod integral, so we can apply the Malliavin integration by part formula on the hybrid term, we obtain :

$$\mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right] e^{-\int_0^T r_u du} 1_{S_T^{s_{ini}}>K}\right) = \mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} D_s^B \left(e^{-\int_0^T r_u du} 1_{S_T^{s_{ini}}>K}\right) ds\right]\right)$$

By applying Malliavin calculus rules, we get :

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right] e^{-\int_0^T r_u du} 1_{S_T^{s_{ini}}>K}\right) &= \mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du}\left(D_s^B\left(1_{S_T^{s_{ini}}>K}\right) - 1_{S_T^{s_{ini}}>K} D_s^B\left(\int_0^T r_u du\right)\right) ds\right]\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du}\left(D_s^B\left(1_{X_T^{s_{ini}}>x}\right) - 1_{S_T^{s_{ini}}>K}\int_s^T D_s^B(r_u)\, du\right) ds\right]\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du}\left(D_s^B\left(X_T^{s_{ini}}\right)\delta_x(X_T^{s_{ini}}) - 1_{S_T^{s_{ini}}>K}\int_s^T \gamma_{s,u} du\right) ds\right]\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\left[\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du}\left(D_s^B\left(X_T^{s_{ini}}\right)\delta_x(X_T^{s_{ini}}) ds - 1_{S_T^{s_{ini}}>K}\Gamma_{s,T}\right) ds\right]\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du} D_s^B\left(X_T^{s_{ini}}\right)\delta_x(X_T^{s_{ini}}) ds\right) - \\
&\qquad \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du} 1_{S_T^{s_{ini}}>K}\int_0^T \gamma_{s,T}\Gamma_{s,T} ds\right)
\end{aligned}$$

Where:

- $X_t^{s_{ini}} = ln\left(\frac{S_t B(0,t)}{s_{ini}}\right)$
- $\Gamma_{t,T} = \int_t^T \gamma_{t,u} du$

In the following, we will use either $X_t^{s_{ini}}$ or X_t.
We denote :

- **term 1** $= 2\frac{\mathbb{E}^{\mathbb{Q}}\left(\int_0^T \gamma_{s,T} e^{-\int_0^T r_u du} D_s^B\left(X_T^{s_{ini}}\right)\delta_x(X_T^{s_{ini}}) ds\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(S_T^{s_{ini}})\right)}$
- **term 2** $= 2\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du} 1_{S_T^{s_{ini}}>K}\int_0^T \gamma_{s,T}\Gamma_{s,T} ds\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(S_T^{s_{ini}})\right)}$

The term 1

Lemma 1 *The Malliavin derivative :* $D_s^B\left(X_T^{s_{ini}}\right)$ *can be computed by a closed formula:*

$$D_t^B\left(X_T^{s_{ini}}\right) = \Gamma_{t,T} + \rho\sigma(T, X_T) + \int_t^T d\epsilon_u \tag{D.11}$$

Where :

$d\epsilon_u = \left(dW_u^{\mathbb{Q}} - \sigma(u, X_u)du\right) D_t^B\left(\sigma(u, X_u)\right) - \rho d\sigma(u, X_u)$

Proof 2 *By direct computation :*

$X_t^{s_{ini}} = \frac{1}{2}\left(\int_0^t \left(\Gamma_{s,t}^2 - \sigma^2(s, X_s)\right) ds\right) + \int_0^t \left(\sigma(s, X_s)dW_s^{\mathbb{Q}} + \Gamma_{s,t}dB_s^{\mathbb{Q}}\right)$

Where $W^{\mathbb{Q}}$ is the risk neutral equity brownian motion.

By applying the Mallavin operator and malliavin operations :

$D_t^B\left(X_T^{s_{ini}}\right) = \Gamma_{t,T} - \int_t^T \sigma(u, X_u)D_t^B\left(\sigma(u, X_u)\right) du + \rho\int_t^T D_t^B\left(\sigma(u, X_u)\right) dB_u^{\mathbb{Q}} + \rho\sigma(t, X_t)$

Thus:

$D_t^B\left(X_T^{s_{ini}}\right) = \Gamma_{t,T} + \rho\sigma(T, X_T) + \int_t^T d\epsilon_u$

We denote :

$$\tilde{\sigma}^2(T, K) = 2\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_x(X_T^{s_{ini}})\int_s^T d\epsilon_u\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(S_T^{s_{ini}})\right)}ds \tag{D.12}$$

Thus, we have :

$$\begin{aligned}
\textbf{term 1} &= \tilde{\sigma}^2(T, K) + 2\frac{\mathbb{E}^{\mathbb{Q}}\left(\int_0^T \gamma_{s,T}e^{-\int_0^T r_u du}\left(\Gamma_{s,T}+\rho\sigma(T,X_T)\right)\delta_x(X_T^{s_{ini}})ds\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(S_T^{s_{ini}})\right)} \\
&= \tilde{\sigma}^2(T, K) + 2\int_0^T \frac{\mathbb{E}^{\mathbb{Q}}\left(\gamma_{s,T}e^{-\int_0^T r_u du}\left(\Gamma_{s,T}+\rho\sigma(T,X_T)\right)\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_x(X_T^{s_{ini}})\right)}ds \\
&= \tilde{\sigma}^2(T, K) + 2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}(\sigma(T,X_T))\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(X_T^{s_{ini}})\right)}ds + 2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds
\end{aligned}$$

Thus, the term $2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}(\sigma(T,X_T))\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(X_T^{s_{ini}})\right)}ds$ is analoguous to an amnesic term that can be written :

$2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}(\sigma(T,X_T))\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(X_T^{s_{ini}})\right)}ds = 2\rho\sigma(T, x)\Gamma_T$

As a result :

$$\textbf{term 1} = \tilde{\sigma}^2(T,K) + 2\rho\sigma(T,x)\Gamma_T + 2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds \tag{D.13}$$

The term 2

This term is actually easier than the previous.
We have :

$$\begin{aligned}\textbf{term 2} &= 2\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}1_{S_T^{sini}>K}\int_0^T \gamma_{s,T}\Gamma_{s,T}ds\right)}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_u du}\delta_K(S_T^{sini})\right)} \\ &= \frac{2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds}{K}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}1_{S_T^{sini}>K}\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{sini})\right)}\end{aligned}$$

As a result, the numerator and denominator can be identified directly with call derivatives with respect to the strike :

$$\textbf{term 2} = \frac{2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds}{K}\left(\frac{-\partial_K C(T,K)}{\partial_{KK}C(T,K)}\right) \tag{D.14}$$

D.2.4 Hybrid local volatility formula

By assembling all the previous terms together, we get :

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \tilde{\sigma}^2(T,K) - 2\rho\sigma(T,x)\Gamma_T - 2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds - \frac{2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds}{K}\left(\frac{\partial_K C(T,K)}{\partial_{KK}C(T,K)}\right) + \Gamma_T^2\frac{\partial_K C(T,K)}{K\partial_{KK}C(T,K)} \tag{D.15}$$

There is an additional property of the Hull & White model (Hull-White model time-homogeneity):

Lemma 2 *We claim that :*

$$2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds = \Gamma_T^2 \tag{D.16}$$

Proof 3 *The proof is obtained by direct computations.*

As a result, we obtain a significant simplification :

$$\sigma^2(T,K) = \sigma^2_{det}(T,K) - \tilde{\sigma}^2(T,K) - 2\rho\sigma(T,x)\Gamma_T - 2\int_0^T \gamma_{s,T}\Gamma_{s,T}ds \tag{D.17}$$

D.3 Fixed point formulation - $\tilde{\sigma}$

In order to obtain the fixed point complete formulation, we must express $\tilde{\sigma}^2(T, K)$ as a function of the hybrid local volatility.
We have by definition :

$$\tilde{\sigma}^2(T,K) = \frac{2}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \int_0^T \gamma_{t,T}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\int_t^T d\epsilon_u\right)dt \tag{D.18}$$

With, $d\epsilon_u = \left(dW_u^{\mathbb{Q}} - \sigma(u, X_u)du\right) D_t^B\left(\sigma(u, X_u)\right) - \rho d\sigma(u, X_u)$

By pushing t close to maturity, we can assume that :

$$\int_0^T \gamma_{t,T}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\int_t^T d\epsilon_u\right)dt \approx -\rho\gamma\tau^2\left(1-\left(1+\frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\frac{d\epsilon_T}{dT}\right) \tag{D.19}$$

Where $\frac{d(.)}{dT}$ is the total classical derivative operator.

The Malliavin derivative on the first term of $d\epsilon$ is computed by a closed formula when t is close to T:

$D_{T_-}^B\left(X_T^{s_{ini}}\right) = \rho\sigma(T, X_T)$

So we apply the Itô formula on $(\sigma(T, X_T))_T$:

$d\sigma(T, X_T) = \frac{\partial\sigma(T,X_T)}{\partial T}dT + \frac{\partial\sigma(T,X_T)}{\partial x}dX_T + \frac{1}{2}\frac{\partial^2\sigma(T,X_T)}{\partial x^2}d\langle X\rangle_T$

We can express the risk neutral dynamic of X_t as such :

$$dX_T = \left(f(0,T) + \left(\int_0^T \gamma_{s,T}dB_s^{\mathbb{Q}}\right) + \int_0^T \gamma_{s,T}\Gamma_{s,T}ds - \frac{1}{2}\sigma_X^2(X_T,T)\right)dT + \sigma_X(X_T,T)dW_T^{\mathbb{Q}} \tag{D.20}$$

We inject the previous expression, we get :

$d\sigma(T, X_T) = \frac{\partial\sigma(T,X_T)}{\partial T}dT + \frac{\partial\sigma(T,X_T)}{\partial x}\left(\left(f(0,T) + \left(\int_0^T \gamma_{s,T}dB_s^{\mathbb{Q}}\right) + \int_0^T \gamma_{s,T}\Gamma_{s,T}ds - \frac{1}{2}\sigma_X^2(X_T,T)\right)dT + \sigma_X(X_T,T)dW\right.$
$\frac{1}{2}\frac{\partial^2\sigma(T,X_T)}{\partial x^2}d\langle X\rangle_T$

On the other hand, according to (17), we have by definition :

$\langle X\rangle_T = \int_0^T \sigma(t, X_t)^2 dt$ (Keep in mind that t is close to T)

By direct differentiation, we have :

$\frac{d\langle X\rangle_T}{dT} = \sigma^2(X_T, T)$

We inject the quadratic variation and we get :

$$d\sigma(T,X_T) = \frac{\partial \sigma(T,X_T)}{\partial T}dT + \frac{\partial \sigma(T,X_T)}{\partial x}\left(f(0,T) + \left(\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right) + \int_0^T \gamma_{s,T}\Gamma_{s,T} ds - \frac{1}{2}\sigma_X^2(X_T,T)\right)dT + \sigma_X(X_T,T)\frac{\partial \sigma(T,X_T)}{\partial x}dW_T^{\mathbb{Q}} + \frac{1}{2}\frac{\partial^2 \sigma(T,X_T)}{\partial x^2}\sigma^2(T,X_T)dT \tag{D.21}$$

We aggregate all the terms to have :

$$d\epsilon_T = \left(dW_T^{\mathbb{Q}} - \sigma(T,X_T)dT\right) D_{T^-}^B\left(\sigma(T,X_T)\right) - \rho\left[\frac{\partial \sigma(T,X_T)}{\partial T}dT + \frac{\partial \sigma(T,X_T)}{\partial x}\left(f(0,T) + \left(\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right) + \int_0^T \gamma_{s,T}\Gamma_{s,T} ds - \frac{1}{2}\sigma_X^2(X_T,T)dT + \sigma_X(X_T,T)\frac{\partial \sigma(T,X_T)}{\partial x}dW_T^{\mathbb{Q}} + \frac{1}{2}\frac{\partial^2 \sigma(T,X_T)}{\partial x^2}\sigma^2(T,X_T)dT \tag{D.22}$$

Which means that :

$$\frac{d\epsilon_T}{dT} = -\rho\left[f(0,T)\frac{\partial \sigma(T,X_T)}{\partial x} + \frac{\partial \sigma(T,X_T)}{\partial T} + \frac{\sigma^2(T,X_T)}{2}\left(\frac{\partial^2 \sigma(T,X_T)}{\partial x^2} + \frac{\partial \sigma(T,X_T)}{\partial x}\right)\right] - \rho\frac{\partial \sigma(T,X_T)}{\partial x}\left(\left(\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right) + \int_0^T \gamma_{s,T}\Gamma_{s,T} ds\right) \tag{D.23}$$

We obtain again the approximation of $\tilde{\sigma}^2(T,K)$ via the formula :

$$\frac{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)}{2\rho\gamma\tau^2\left(1-\left(1+\frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)}\tilde{\sigma}^2(T,K) \approx \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)\rho\left[\left[f(0,T)\frac{\partial \sigma(T,X_T)}{\partial x} + \frac{\partial \sigma(T,x)}{\partial T} + \frac{\sigma^2(T,x)}{2}\left(\frac{\partial^2 \sigma(T,x)}{\partial x^2} + \frac{\partial \sigma(T,x)}{\partial x}\right) + \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)\int_0^T \gamma_{s,T}\Gamma_{s,T} ds\frac{\partial \sigma(X_T,T)}{\partial X} + \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)\frac{\partial \sigma(x,T)}{\partial x}\frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{D.24}$$

Actually, we can neglect the effect of the term: $\int_0^T \gamma_{s,T}\Gamma_{s,T} ds$ using the amnesic approximation.

We have the following lemma :

Lemma 3 *The relation between $\delta_x(X_T^{s_{ini}})$ and $\delta_K(S_T^{s_{ini}})$ is proportional :*

$$\delta_K(S_T^{s_{ini}}) = K\delta_x(X_T^{s_{ini}}) \tag{D.25}$$

Proof 4 *The proof is obtained by applying the chain rule derivation.*

Thus, we obtain :

$$\tilde{\sigma}^2(T,K) \approx 2\rho^2\gamma\tau^2\left(1-\left(1+\frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)\left[\Lambda(x,T) + \Phi(x,T)\right] \tag{D.26}$$

Where :

$$\sigma_X := \sigma \tag{D.27}$$

$$\Lambda(x,T) = \frac{\sigma_X^2(x,T)\left[\frac{\partial \sigma_X(x,T)}{\partial x} + \frac{\partial^2 \sigma_X(x,T)}{\partial x^2}\right]}{2} + \frac{\partial \sigma_X(x,T)}{\partial t} + f(0,T)\frac{\partial \sigma(T,X_T)}{\partial x} \tag{D.28}$$

and

$$\Phi(x,T) = \frac{\partial \sigma_X(x,T)}{\partial x} \frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{D.29}$$

Thus, using the scale (T,x), we summarise all the key formulas.

D.4 Summary

$$\sigma^2(x,T) = \sigma_{det}^2(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(x,T)\Gamma_T - \Gamma_T^2 \tag{D.30}$$

$$\sigma_{det}^2(T,K) = \frac{\partial_T C(T,K) - K f(0,T)\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \tag{D.31}$$

$$\tilde{\sigma}^2(T,x) = \frac{2}{K\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_K(S_T^{s_{ini}})\right)} \int_0^T \gamma_{t,T}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\int_t^T d\epsilon_u\right)dt \tag{D.32}$$

$$\tilde{\sigma}^2(T,x) \approx 2\rho^2\gamma\tau^2\left(1 - \left(1 + \frac{T}{\tau}\right)e^{-\frac{T}{\tau}}\right)\left[\Lambda(x,T) + \Phi(x,T)\right] \tag{D.33}$$

Where :

$$\Lambda(x,T) = \frac{\sigma^2(x,T)\left[\frac{\partial \sigma(x,T)}{\partial X} + \frac{\partial^2 \sigma(x,T)}{\partial X^2}\right]}{2} + \frac{\partial \sigma(x,T)}{\partial t} + f(0,T)\frac{\partial \sigma(x,T)}{\partial x} \tag{D.34}$$

and

$$\Phi(x,T) = \frac{\partial \sigma(x,T)}{\partial x} \frac{\mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right]\right\}}{\mathbb{E}_T\left\{\delta_x(X_T)\right\}} \tag{D.35}$$

$$\hat{\tilde{\sigma}}^2(x,T) = \tilde{\sigma}^2(T,x) + 2\rho\sigma(x,T)\Gamma_T \tag{D.36}$$

$$\zeta(x,T) \triangleq \mathbb{E}_T\left\{\delta_x(X_T)\left[\int_0^T \gamma_{s,T} dB_s^{\mathbb{Q}}\right]\right\} \tag{D.37}$$

$$= \int_0^T \gamma_{s,T}\mathbb{E}_T \left\{ \frac{\partial \delta_x(X_T)}{\partial dB_s^{\mathbb{Q}}} \right\}$$

$$= \int_0^T \gamma_{s,T}\mathbb{E}_T \left\{ \delta'_x(X_T) \frac{\partial X_T}{\partial dB_s^{\mathbb{Q}}} \right\}$$

$$\sim \Gamma_T \mathbb{E}_T \left\{ \delta'_x(X_T) \frac{\partial X_T}{\partial dB_T^{\mathbb{Q}}} \right\}$$

$$\sim \Gamma_T \rho \mathbb{E}_T \left\{ \delta'_x(X_T)\sigma(X_T, T) \right\}$$

As:

$$\mathbb{E}_T \left\{ \delta'_x(X_T)\sigma(X_T, T) \right\} = \int_{y\in\mathbb{R}} \delta'_x(y)\sigma(y,T)p_X(y,T)dy$$

$$= -\int_{y\in\mathbb{R}} \delta_x(y) \frac{\partial}{\partial y} \left(\sigma(y,T)p_X(y,T) \right) dy$$

$$= -\left(\frac{\partial p_X(x,T)}{\partial X}\sigma(x,t) + \frac{\partial \sigma(x,T)}{\partial X} p_X(x,t) \right)$$

with $p_X(x,t) = \mathbb{E}_T \{\delta_x(X_T)\}$ And finally:

$$\Phi(x,T) \sim -\rho\Gamma_T \frac{\partial \sigma(x,T)}{\partial x} \left(\frac{\partial \sigma(x,T)}{\partial x} + \sigma(x,T) \frac{\partial \ln(p_X(x,T))}{\partial x} \right) \tag{D.38}$$

D.5 Fixed point algorithm

We recall the same fixed point algorithm as in the section (3.4.4), as the results are exactly the same.

Indeed, the algorithm is:

$$\sigma_{n+1}^2 = F(\sigma_n), n \in \mathbb{N} \tag{D.39}$$

Where F is the following differential operator:

$$F(\sigma) = \sigma_{det}^2(T,K) - \tilde{\sigma}^2(T,K) - 2\int_0^T \gamma_{s,T}\Gamma_{t,T}dt - 2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\left(\sigma(T,X_T)\right)\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)}ds \tag{D.40}$$

each of the terms :

- $\tilde{\sigma}^2(T,K)$
- $2\rho\int_0^T \gamma_{s,T}\frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}(\sigma(T,X_T))\delta_x(X_T^{s_{ini}})\right)}{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\delta_x(X_T^{s_{ini}})\right)}ds$

depend on the local volatility.

Here is the algorithm in practice :

Input: An initial local volatility surface, a number of maximum number of iterations N,the current number of iterations n, an error ϵ
Output: Hybrid local volatility surface

1. **Initialisation** :
   - Set σ_0 to be the initial local volatility considered
   - n=0
2. **Iteration n**
   While n<N and $norm(\sigma_{n+1} - \sigma_n) > \epsilon$:
   - $\sigma_n = \sigma_{n+1}$,
   - $\sigma_{n+1} = F(\sigma_n)$
3. **Refresh** : n=n+1

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