# Local volatility models under stochastic interest rates

## Final-term presentation and synthesis

Othmane ZARHALI

Finastra - Quantitative research

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# Content

# Overview

Let's consider a one dimensional underlying asset with the following risk neutral dynamic:

$$\frac{dS_t^s}{S_t^s} = rdt + \sigma dW_t^{\mathbb{Q}} \tag{1}$$

The scenarios that are of interest for us are:

- Both r and $\sigma$ constant : closed Black and Scholes call and put formulas, But volatility surface is flat
- r constant and $\sigma$ deterministic time dependent : closed Black and Scholes call and put formulas applied with volatility:

$$\sigma_{BS}^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt \tag{2}$$

# Overview

- r stochastic and $\sigma$ deterministic time dependent : This is the case of a flat local volatility with stochastic interest rates that will be studied first.
- r stochastic and $\sigma$ time and space dependent : This is the case of a true hybrid local volatility surface with stochastic interest rates.

The aim is to build a bridge between the **hybrid local volatility** with stochastic rates and the **Dupire's formula (with deterministic rates)**

# Introduction of the local volatility model

It's well known that the local volatility is so that the risk neutral dynamic of the underlying asset is :

$$\frac{dS_t^s}{S_t^s} = rdt + \sigma(t, S_t^s)dW_t^{\mathbb{Q}} \tag{3}$$

# The local volatility hybrid model

Another issue is to try to fit more market features by considering **stochastic interest rates** :

$$\frac{dS_t^s}{S_t^s} = r_t dt + \sigma(t, S_t^s) dW_t^{\mathbb{Q}} \tag{4}$$

And also to try to handle the disadvantage of the risk neutral asset tail by adding an **additive stochastic noise** on interest rates.

Which leads to an hybrid model of a one dimensional asset with stochastic interest rate.

# Local volatility hybrid model

## Toy model: Flat local volatility surface

This is the toy flat local volatility model :

$$\begin{cases} \frac{dS_t^s}{S_t^s} = r_t dt + \sigma(t) dW_t^{\mathbb{Q}} \\ r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}} \\ d\left\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \right\rangle_t = \rho_t dt \end{cases} \tag{5}$$

In this context, we have the static replication equation:

$$\forall T \geq 0, \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}} \left[ log \left( \frac{S_T^s}{\frac{s}{B(0,T)}} \right) \right] = -\frac{1}{2} \int_0^T \sigma_{eq}^2(\theta) d\theta \tag{6}$$

Where:

$$\sigma_{eq}^2(\theta) = \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta \sigma(\theta) \Gamma_{\theta,T} \tag{7}$$

# Local volatility hybrid model

## Toy model: Flat local volatility surface

And:

$$\Gamma_{\theta,t} = \int_{\theta}^{t} \gamma_{\theta,s} ds \tag{8}$$

And $\mathbb{Q}^{\mathbb{T}}$ is the forward neutral probability measure with maturity $T$.
In particular for a fixed maturity :

$$\mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[ log\left( \frac{S_T^s}{\frac{s}{B(0,T)}} \right) \right] = -\frac{1}{2}\int_0^T \sigma_{eq}^2(\theta) d\theta \tag{9}$$

# Local volatility hybrid model

## Toy model: Flat local volatility surface

Then:

$$\frac{\partial \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[log\left(\frac{S^s_T}{\frac{s}{B(0,T)}}\right)\right]}{\partial T} = -\frac{1}{2}\left(\sigma^2_{eq}(T) + \int_0^T 2\sigma_{eq}(\theta)\frac{\partial \sigma_{eq}(\theta)}{\partial T}d\theta\right) \quad (10)$$

# Local volatility hybrid model

## Toy model: Flat local volatility surface

As a consequence:

- (10) is an ODE in $\sigma(T) = \sigma_{eq}(T)$ that can be solved iteratively and the left hand side can be replicated by Carr-Madan formula.
- for small time to maturities, we can neglect the deterministic integral of the right hand side and we have :

$$\frac{\partial \mathbb{E}^{\mathbb{Q}^{\mathbb{T}}}\left[ log\left( \frac{S_T^s}{\frac{s}{B(0,T)}} \right) \right]}{\partial T} \approx -\frac{1}{2}\sigma_{eq}^2(T) \tag{11}$$

Then, we obtain a flat local volatility calibrated to market data.

# Local volatility hybrid model

## Toy model: Flat local volatility surface - stochastic effect of rates

The aim is to **quantify the impact of the stochastic aspect of interest rates** on the hybrid local volatility.
The implied theoretical volatility is:

$$\sigma_{th}^2(T) = \frac{1}{T}\int_0^T \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta\sigma(\theta)\Gamma_{\theta,T}d\theta \qquad (12)$$

It's similar to consider the implied theoretical variance :

$$V_{th}(T) = \int_0^T \sigma^2(\theta) + \Gamma_{\theta,T}^2 + 2\rho_\theta\sigma(\theta)\Gamma_{\theta,T}d\theta \qquad (13)$$

P.S : The correlation factor $\rho$ will be constant

# Local volatility hybrid model

## Toy model: Flat local volatility surface - stochastic effect of rates

In the specific case of Hull&White model with constant equity-rate correlation coefficient :

$$dr_t = (\theta - \alpha r_t)dt + \gamma dB_t^{\mathbb{Q}} \tag{14}$$

We obtain :

$$\sigma^2(T) = \frac{dV_{th}(T)}{dT} - \Gamma_{0,T}^2 - 2\rho\gamma \int_0^T e^{-\alpha(T-u)}\sigma(u)du \tag{15}$$

Which is a non linear ODE that can be resolved by:

- Identification of the term $\frac{dV_{th}(T)}{dT}$ with market data
- Choosing our favorite numerical method (finite differences for instance).

# Local volatility hybrid model

## Toy model: Flat local volatility surface - stochastic effect of rates

We can simplify the previous ODE by considering a resolution for large time to maturities. We have in the case of H&W :

$$\sigma_\infty = \sqrt{\frac{dV_{th}(T)}{dT}|T\infty - (1-\rho^2)\left(\frac{\gamma}{\alpha}\right)^2} - \rho\frac{\gamma}{\alpha} \qquad (16)$$

We denote: $\sigma_\infty^{det^2} = \frac{dV_{th}(T)}{dT}|T\infty$ we then obtain the formula:

$$\boxed{\sigma_\infty = \sqrt{\sigma_\infty^{det^2} - (1-\rho^2)\left(\frac{\gamma}{\alpha}\right)^2} - \rho\frac{\gamma}{\alpha}} \qquad (17)$$

# Local volatility hybrid model

## Toy model: Flat local volatility surface - stochastic effect of rates

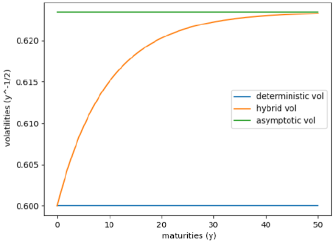
It's worth telling from the equation (17) that:

- The impact of the stochasticity of interest rates on the flat local volatility is quantified the additional term $\rho\frac{\gamma}{\alpha}$.
- (17) is well defined, as for a vanishing $\gamma$, the flat hybrid local volatility is nothing but a Dupire's one, which is expected.
- We can verify (17) numerically:

# Local volatility hybrid model

## Toy model: Flat local volatility surface - stochastic effect of rates

For a flat $T \longrightarrow \sigma_T^{det^2}$, we obtain:




ODE calibration for flat local volatility surface

This, actually highlights the hybrid local volatility adjustment with respect to the Deterministic (Dupire) one.

# Local volatility hybrid model

## Toy model: Flat local volatility surface - Conclusion

Here are some other additional qualitative remarks:

- The maturity is a key element of the adjustment quantification.
- The adjustment term is **model rate dependent** and we can at least pretend be hedged from the correlation equity-rate but the rate risk model is not vanished.
- The aim of the flat local volatility toy model is to figure out some elementary features of the hybrid local volatility adjustment, and **to have an initial guess of what to expect in the general hybrid local volatility case (non flat)**.
- The adjustment is finally the element of interest .

# Non flat local volatility calibration

By direct computations, we obtain the hybrid local volatility model in the form :

$$\sigma^2(T,K) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} - \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds}1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \quad (18)$$

Where this adjustment term that is due to the rates's stochasticity is :

$$Adj(T) = \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds}1_{S_T^{s_{ini}}>K}\right)}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \quad (19)$$

It depends on :

- The joint distribution of the equity and rate
- The probability measure of the previous joint distribution

# Non flat local volatility calibration

Thus, the hybrid term $\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)$ will be handled using two approaches:

- **A generic approach** : Brute Monte Carlo computation under the risk neutral probability space with eventually some boosting schemes (Richardson Romberg extrapolation for the Monte Carlo boosting).
- **A specific approach in the case of gaussian rates** : PDE (or iterative PDE) obtained by Malliavin integration by part formula.

# Non flat local volatility calibration

## Generic approach : Monte Carlo scheme

We have the following weak approximation:

$$\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}}>K}\right) \approx \frac{1}{M}\sum_{i=1}^{M}\left(r_T^{(i)} e^{-\delta \sum_{k=1}^{N} r_{t_k}^{(i)}} 1_{S_T^{(i)}>K}\right) \tag{20}$$

- $\delta$ is the time step discretisation
- $M$ the length of the Monte Carlo sample
- $N$ the length of the time discretisation of $[0, T]$

Indeed, the Euler scheme of the model is the following :

$$\begin{cases} S_{t_{k+1}}^{s_{ini}} = S_{t_k}^{s_{ini}}\left(1 + r_{t_k}\delta\right) + S_{t_k}^{s_{ini}}\sigma(t_k, S_{t_k}^{s_{ini}})\sqrt{\delta}G \\ r_{t_{k+1}} = r_{t_k} + b(t_k, r_{t_k})\delta + \tilde{\sigma}(t_k, r_{t_k})\sqrt{\delta}\left(\rho G + \sqrt{1-\rho^2}\tilde{G}\right) \end{cases} \tag{21}$$

# Non flat local volatility calibration

## Specific approach : "PDE expansion"

We denote the explicit spot rate as previously :

$$r_t = \mu_t + \int_0^t \gamma_{s,t} dB_s^{\mathbb{Q}} \qquad (22)$$

In the following we will expose the iterated PDE method.

# Non flat local volatility calibration

## Specific approach : "PDE expansion"

We obtain the following results:

$$\sigma^2(T,x) = \sigma^2_{det}(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(T,x)\Gamma_T - \Gamma_T^2 \quad (23)$$

$$\tilde{\sigma}^2(T,x) \approx 2\rho^2\gamma\tau^2 \left(1 - \left(1 + \frac{T}{\tau}\right) e^{-\frac{T}{\tau}}\right) [\Lambda(x,T) + \Phi(x,T)] \quad (24)$$

Where:

$$\Lambda(x,T) = \frac{\sigma_X^2(T,x)\left[\frac{\partial\sigma(T,x)}{\partial X} + \frac{\partial^2\sigma(T,x)}{\partial X^2}\right]}{2} + \frac{\partial\sigma(T,x)}{\partial t} + f(0,T)\frac{\partial\sigma(T,x)}{\partial x} \quad (25)$$

# Non flat local volatility calibration

## Specific approach : "PDE expansion"

And on the other hand :

$$\Phi(x, T) \sim -\rho\Gamma_T \frac{\partial\sigma(T,x)}{\partial x}\left(\frac{\partial\sigma(T,x)}{\partial x} + \sigma(T,x)\frac{\partial \ln(p_X(x,T))}{\partial x}\right) \tag{26}$$

Where $p_X(x, T)$ is the density of the log-moneyness.
As a result, we have the following remarks:

- We claim that the term $-\tilde{\sigma}^2(T,K) - 2\rho\sigma(T,x)\Gamma_T - \Gamma_T^2$ is assumed to be small as it is just an adjustment of the deterministic local volatility.

# Non flat local volatility calibration

## Specific approach : "PDE expansion"

In fact, for classical Hull & White parameters, we have:

$$r_t = r_0 e^{-\alpha t} + \theta \left(1 - e^{-\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-s)} dB_s^{\mathbb{Q}} \tag{27}$$

With : $\sigma \approx 1e-2$.As $\Gamma_T \propto \sigma$, thus $\Gamma_T^2 \propto \sigma^2 \approx 1e-4$. Furthermore, After computations, we have $\tilde{\sigma} \approx 1e-7$ for small maturities. Then we can conclude that the previous term is nothing but a correction (This will be highlighted int he numerical simulations).

# Non flat local volatility calibration

## Specific approach : "PDE expansion"

- In order to have a meaningful expansion of the hybrid local volatility, it should be able to reproduce classical effects:
  - Obtain the Dupire's local volatility in the case of vanishing volatility rates
  - A significant simplification in the case of low hybrid local volatility skew
  - An analogy with the toy flat local volatility model

Which is all the case.

# Numerical deployment

## Numerical evidence of the impact of volatility rates

We recall the following key formula:

$$\sigma^2(T,K) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} - \frac{K\mathbb{E}^{\mathbb{Q}}\left(r_T e^{-\int_0^T r_s ds} 1_{S_T^{s_{ini}} > K}\right)}{\frac{1}{2}K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \quad (28)$$

The aim of these first simulations is to study the impact of the volatility rates in the case of the H&W model:

- We consider a fixed hybrid local volatility
- We study the impact of the volatility rate in the equity smile, which is encompassed in $\sigma^2_{det}(T,K)$.

# Numerical deployment

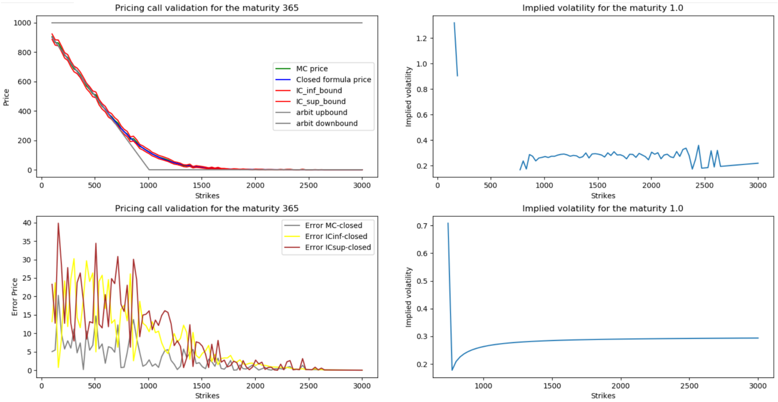
## Numerical evidence of the impact of volatility rates

The parameters that we vary in each numerical scenario in the following tests are :

- **The hybrid local volatility function**
  - A constant surface
- **The rate model parameters**
  - Case 1: Low volatility rate ($\approx 0.001\%$)
  - Case 2: Medium volatility rate ($\approx 10\%$)
  - Case 3: High volatility rate ($\approx 30\%$)

# Numerical deployment

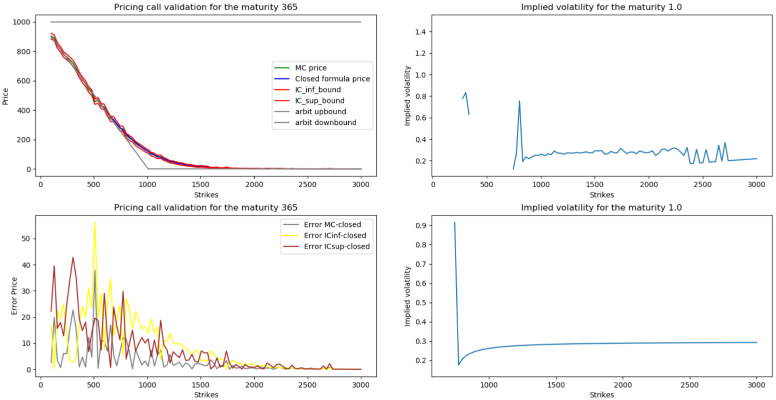
## Numerical evidence of the impact of volatility rates




Low rate volatility

# Numerical deployment

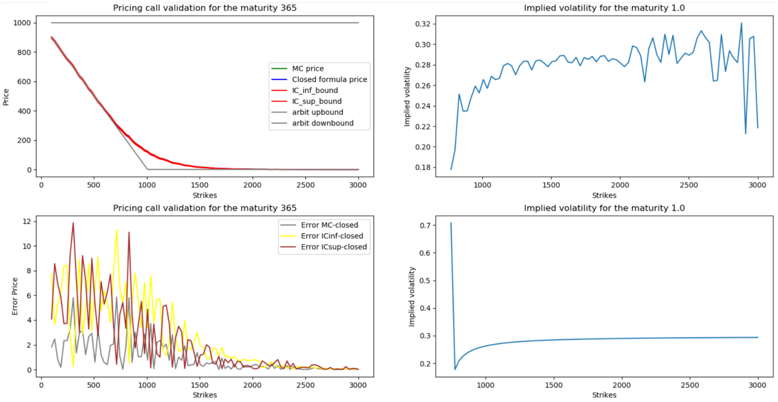
## Numerical evidence of the impact of volatility rates




Medium rate's volatility

# Numerical deployment

## Numerical evidence of the impact of volatility rates




High rate volatility

# Numerical deployment

## Numerical evidence of the impact of volatility rates

We conclude numerically that :

- The rate volatility is in fact a key parameter in the flat local volatility adjustment.
- But also the equity-rate correlation factor has its contribution

Which is inline with the theoretical guesses and the toy flat local volatility model.

# Numerical deployment

## Hybrid local volatility computation - General computation

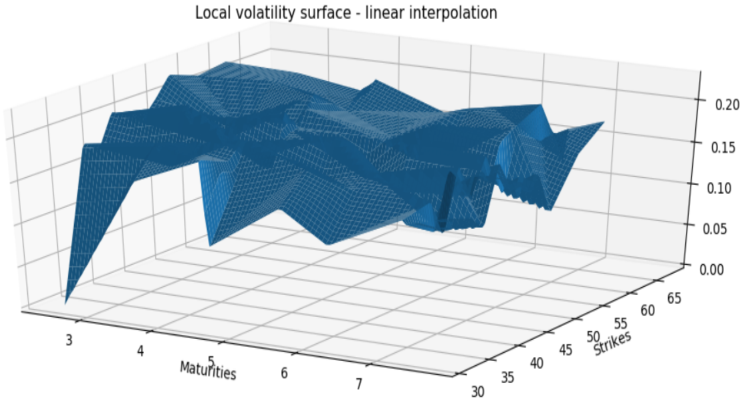
Here is the computation algorithm of the hybrid local volatility surface:

**Input:** Model parameters, Equity smile, Monte Carlo parameters
**Output:** Hybrid local volatility surface

1. Perform interpolation of the Smile by strike and maturity
2. Estimate all the partial derivatives by finite differences
3. Perform the Monte Carlo computation of the hybrid term
4. Agregate all the terms
5. Perform grid interpolation
6. Return the hybrid local volatility surface

# Numerical deployment

## Hybrid local volatility computation - General computation




Local volatility surface with non flat equity smile and high rate vol

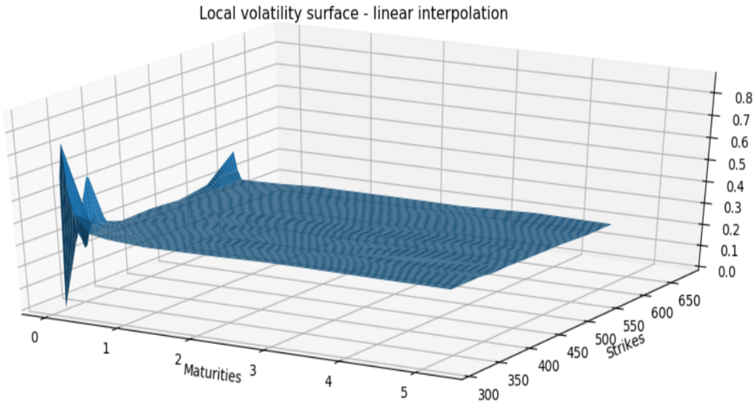
# Numerical deployment

## Hybrid local volatility computation - General computation

1. The impact of the volatility rate on the Hybrid local volatility calibration leads also to a noisy local volatility surface, as we can see in the previous slide.
2. Actually this is not sufficient to see if the hybrid local volatility surface corresponds really to what is expected.
3. Let's be in the typical case of a vanishing volatility rate and a flat equity smile, and to see that this corresponds to the related Dupire's local volatility surface.

# Numerical deployment

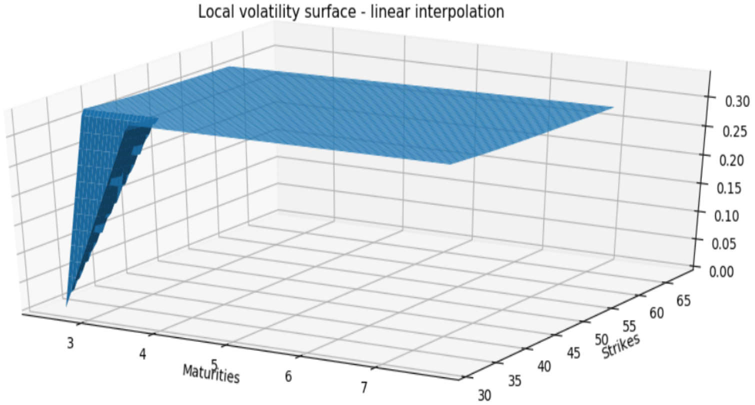
## Hybrid local volatility computation - MC approach




Hybrid local volatility surface with flat equity smile and low rate vol

# Numerical deployment

## Hybrid local volatility computation - MC approach




Dupire's local volatility surface with flat equity smile

# Numerical deployment

## Hybrid local volatility computation - MC approach

- The hybrid local volatility surface represents really **what is expected** :
  - A sensitivity to the volatility rate
  - A sensitivity to the equity-rate correlation
- But still the quantification of the rate's stochasticity is not controlled. Which is the main strength of the hybrid local volatility expansion.

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion"

We recall the main formula :

$$\sigma^2(T,x) = \sigma^2_{det}(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(T,x)\Gamma_T - \Gamma_T^2 \quad (29)$$

$$\tilde{\sigma}^2(T,x) \approx 2\rho^2\gamma\tau^2 \left(1 - \left(1 + \frac{T}{\tau}\right) e^{-\frac{T}{\tau}}\right) [\Lambda(x, T) + \Phi(x, T)] \quad (30)$$

It appears from the formulation that the hybrid local volatility skew dependence and the rate model dependence are quit separated.

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion"

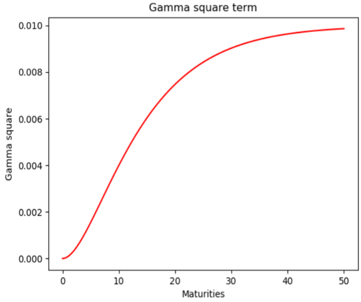
There is also a number of relevant remarks:

- The formula allows to control the hybrid local volatility calibration by the adjustment of the rate's parameters and the skew terms
- The previous formula is symmetric: given the deterministic local volatility we can determine the hybrid local volatility and vice-verca.

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Terms analysis

- Actually, $\sigma^2_{det}(T, x)$ is non compressible.
- The term $\Gamma^2_T$ looks like for a volatility rate 1%:

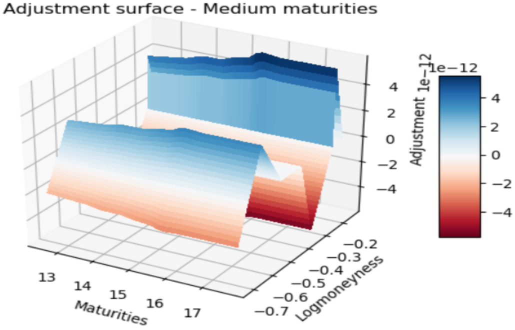



Gamma square

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Terms analysis

- The adjustment $\tilde{\sigma}^2(T, x)$ term depend on the hybrid local volatility's regularity:

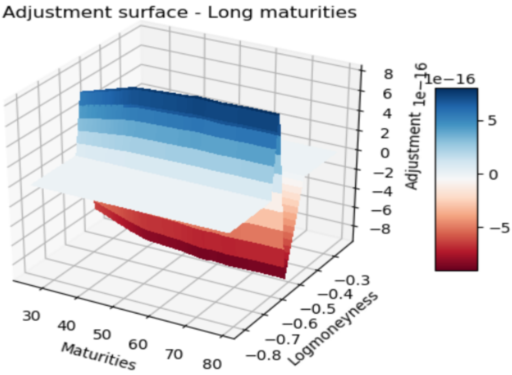



Adjustment term

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Terms analysis

- The adjustment vanishes for long maturities:




Adjustment term

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

The expansion formula is of the form:

$$\sigma^2 = F(\sigma) \qquad (31)$$

Where F is the following differential operator:

$$F(\sigma) = \sigma_{det}^2(T,x) - \tilde{\sigma}^2(T,x) - 2\rho\sigma(T,x)\Gamma_T - \Gamma_T^2 \qquad (32)$$

Thus, the idea is to enroll a sort of a fixed point algorithm:

$$\begin{cases} \sigma_{n+1}^2 = F(\sigma_n), n \in \mathbb{N} \\ \sigma_0 \in \mathcal{M}(p,q) \end{cases} \qquad (33)$$

Where $p$ and $q$ are respectively the number of maturities and logmoneynesses.

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

As we have no idea of the structure of the differential operator (Lipschitz,...), the aim is to

- Start with a not bad hybrid local volatility
- Correct it in 2 or 3 iterations maximum

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

Let's find out if the classical stylized facts are reproduced by the method.

We consider the following scenarios :

- **Scenario 1**: Vanishing correlation and volatility rate vs flat hybrid local volatility
- **Scenario 2**: Vanishing correlation and volatility rate vs non flat hybrid local volatility
- **Scenario 3**: Non vanishing correlation and volatility rate vs flat hybrid local volatility
- **Scenario 4**: Non flat hybrid local volatility

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

Here are the respective expectations and the results:

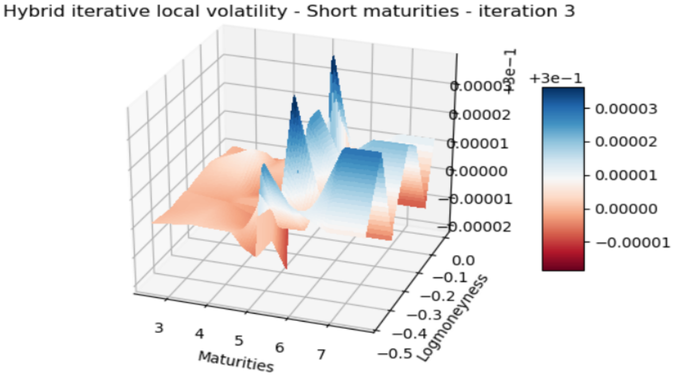
| Scenarios | Expectations | Results |
|---|---|---|
| 1 | The same flat hybrid local volatility | ✓ |
| 2 | A corrected flat hybrid local volatility | ✓ |
| 3 | A corrected hybrid local volatility | ✓ |
| 4 | A corrected non flat hybrid local volatility | ✓ |

# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

Let's present some numerical simulations.

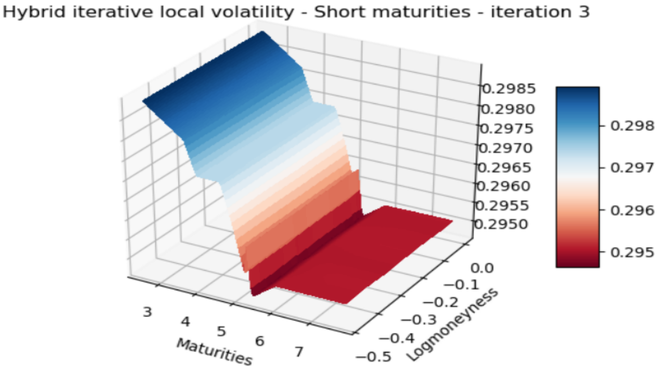
**Scenario 1**:



# Numerical deployment

## Hybrid local volatility computation - "PDE expansion" - Computation in practice

**Scenario 3**:




Output hybrid local volatility

# Conclusion

- The aim of the intern was to quantify the impact of the stochastic interest rates on the local volatility
- The toy flat local volatility model was important to construct the initial guesses
- Some analogies are found out between the flat and non flat local volatility ($\rho$, and $\gamma$ adjustment dependence,...)
- The "PDE expansion" allow to correct iteratively the hybrid local volatility, but some regularity assumptions on the hybrid local volatility are required.

Thanks for your attention